Optimality Conditions of Fractional Order Discontinuous Variational Problems

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Abstract

In this paper, an approach is presented, to construct the optimality conditions of fractional order unconstrained and constrained variational problems with discontinuous functional, having different multi fractional order derivatives, on fixed and moving boundaries.

Keywords: Optimization, Calculus of Variations, Fractional Calculus, Fractional Calculus of variations.

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Introduction

The field of calculus of variations is of significant importance in various disciplines such as science, engineering, and pure and applied mathematics (see, for example, [1-9]. They present a Bliss-type multiplier rule for constrained variational problems with delay. Calculus of variations has been the starting point for various approximate numerical schemes, see [4] & [6].

Fractional derivatives, or more precisely derivatives of arbitrary orders, have played a significant role in engineering, science, and in pure and applied mathematics in recent years. Additional background, survey, and application of this field in science, engineering, and mathematics can be found, among others, in [1], [5] & [7-13].

Recent investigations have shown that many physical systems can be represented more accurately using fractional derivative formulations, see [1]. Given this, one can imagine obtaining these formulations by minimizing certain functional. These functional will naturally contain fractional order derivative, and mathematical tools analogous to calculus of variations will be needed to minimize these functional. However, very little work has been done in the area of fractional calculus of variations, see[2] & [3].

Fractional calculus is a branch of mathematics which deals with the investigation and applications of integrals and derivatives of arbitrary order. Fractional calculus may be considered as old and yet a novel topic, actually, it is an old topic since starting from some spectrum of Leibniz (1695–1697) and Euler (1730) who said "When n is an integer, the ratio $\frac{d^n f}{dx^n}$ can be made if n is fraction?", it has been developed up to nowadays. In fact, the idea of generalizing the notion of derivative to non – integer order, in particular to the order of ½ (which is called semi – integral or semi – derivative) is found in the correspondence of Leibniz and Bernoulli, L'Hopital and Wallis. Euler took the first step by observing that the result of the derivative evaluation of the power function has a meaning for non integer order thanks to his Gamma function [1].

The calculus of variations essentially is an extension of minimizing or maximizing a function of one variable to problems involving minimizing or maximizing a functional. Typically, a functional is an integral whose integrand involves an unknown function and its derivatives; the objective is to find the (not necessarily unique) function that makes the integral stationary within a given class of functions. see [12].

The study of problems of the calculus of variations with fractional derivatives is a rather recent subject, the main result being the fractional necessary optimality condition of Euler – Lagrange to be obtained [14].

Riewe [2], [3] obtained a version of the Euler – Lagrange equations for problem of the Calculus of Variations with fractional derivatives. More recently, Agrawal [10] gave a formulation for variational problems with right and left fractional derivatives in the Riemann – Liouville sense, and constructed the optimality necessary conditions of fractional variational problems on fixed boundaries, with non-fractional constraint.

In this paper, we are using an approximated approach to obtain the optimality conditions for unconstrained and constrained fractional order discontinuous variational problems, on fixed and moving boundaries, with multi-fractional order derivatives.

Basic Concepts

Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an n-fold integral.

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} f(x_n)dx_n = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}dt \qquad \ldots(2.1)$$

By n-fold here means that the integration is deployed n-times. Since $(n-1)!=\Gamma(n)$, Riemann realized that the RHS of (1) might have meaning even when n takes non-integer values. see [13]

Let f be a continuous function on [a, b], for all $x\in[a, b]$. The left (resp. right) Riemann-Liouville derivative at x is given by

$$D_-^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha}}dt,$$

$$D_+^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b \frac{f(t)}{(x-t)^{\alpha}} dt$$

…(2.2)

with (n-1< α <n), and n is positive integer, and also, in [10] f is said to be α-differentiable, if $D_+^{\alpha} = D_-^{\alpha}$ are exists and equal, (denoted by D^{α}).While the Riemann-Liouville Integeral can be considered and denoted by $\left(I^{\alpha} = D^{-\alpha}\right)$.

In constructing the necessary conditions of the fractional variational problems, we are needed the following property, called Classical product rule for RL-derivatives, for all α >0,(see [2] & [3])

$$\int_a^b D_+^{\alpha} f(t) g(t) dt = \left[f(t) g(t)\right]_a^b - \int_a^b f(t) D_-^{\alpha} g(t) dt$$

…(2.3)

Formula (2.3) gives a strong connection between D_+^{α} & D_-^{α} via a generalized integration by part. This relation is very useful in order to apply the following Fundamental Lemma of calculus of variations:

Fundamental Lemma

If a function ϕ(x) is continuous in (x_0, x_1) and if $\int_{x_0}^{x_1} \phi(x)\gamma(x)dx = 0$ **for an arbitrary function γ(x) subject to some conditions of general character only, then $\phi(x) \equiv 0$ along the interval [x_0, x_1].**

Differential equations of variational problems can be integrated easily only in exceptional cases. It is therefore essential to find other methods of solution of these problems. The fundamental idea underlying the so called direct methods is to consider a variational problem as a limit problem for some problem of extrema of a function of a finite number of variables. This problem is solved by usual methods, and then by a kind of limiting process the solution of the original variational problem is obtained.

The functional v(y(x)) can be considered as a function of an infinite set of variables. This is fairly evident, if we assume that the admissible functions can be represented by any series. In order to determine such a function, it is sufficient to determine all the coefficients, so that the value of the functional v(y(x)) is completely determined by an infinite sequence of numbers, i.e. the functional is a function of an infinite set of variables v(y(x))=φ(a_0,a_1,…,a_n,…) Consequently, the difference between variational problems and problems of extrema of functions of finite number of variables is in the number of variables— in variational problems of extrema this number is infinite. Therefore, the fundamental idea of direct methods that consists in considering a variational problem as a limiting case of a problem of extrema of an ordinary function of finite number of variables is of much interest. For more details about direct methods see [8].

The Problem

Unconstraint Problem

We consider the following simplest variational problem:

$$\min v\left(x, y(x), y^{(\alpha)}\right)=\int_{x_s}^{x_f} F\left(x, y, y^{(\alpha)}\right)dx\text{, } \alpha>0\text{, non-integer} \qquad \ldots(3.1)$$

(Fixed Boundaries):

If F is discontinuous on (x_k), for k=1,…,n. Since the fundamental Lemma of the calculus of variation can't be applied, because of the discontinuities, it is more convenient to calculate the value of $v\left(x, y(x), y^{(\alpha)}\right)$ along the polygonal curves approximately, so it is convenient to replace the integral

$$\int_{x_s}^{x_f} F\left(x, y, y^{(\alpha)}\right)dx \cong \int_{x_s}^{x_1-s} F\left(x, y, y^{(\alpha)}\right)dx+\sum_{k=1}^{n-1}\int_{x_k+s}^{x_{k+1}-s} F\left(x, y, y^{(\alpha)}\right)dx+\int_{x_n+s}^{x_f} F\left(x, y, y^{(\alpha)}\right)dx \qquad \ldots(3.2)$$

with prescribed condition on fixed boundary, and

$$\delta y\big|_{x_s}=\delta y\big|_{x_k-s}=\delta y\big|_{x_k+s}=\delta y\big|_{x_f}=0,(k=1,\ldots,n)$$

We are perform the classical steps as in [8] on each sub-interval, for the extrema of (3.1), in which F is continuous in all sub-intervals to obtain

$$\delta v = F_{y^{(\alpha)}}\delta y\Big]_{x_s}^{x_1-s} - \int_{x_s}^{x_1-s}\left(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx + \sum_{k=1}^{n-1}\left\{F_{y^{(\alpha)}}\delta y\Big]_{x_k+s}^{x_{k+1}-s} - \int_{x_k+s}^{x_{k+1}-s}\left(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx\right\}$$

$$+ F_{y^{(\alpha)}}\delta y\Big]_{x_n+s}^{x_f} - \int_{x_n+s}^{x_f}\left(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx = 0$$

The above formula (3.2) can be written as follows

$$\delta v = F_{y^{(\alpha)}}\delta y\Big]_{x_s}^{x_1-s} + \sum_{k=1}^{n-1}\left\{F_{y^{(\alpha)}}\delta y\Big]_{x_k+s}^{x_{k+1}-s}\right\} + F_{y^{(\alpha)}}\delta y\Big]_{x_n+1}^{x_f} - \int_{x_s}^{x_1-s}\left(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx +$$

$$\sum_{k=1}^{n-1}\left\{\int_{x_k+s}^{x_{k+1}-s}\left(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx\right\} + \int_{x_n+s}^{x_f}\left(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx = 0$$

…(3.3)

Since, the first three terms are vanish, and then applying the fundamental lemma, on the integrand of the last three terms, we obtain the following condition

$$\left(F_y - \frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right) = 0$$

…(3.4)

which the necessary condition for (3.1).

(Moving Boundaries):

By recalling (3.1), where F is discontinuous on (x_k), for k=1,…,n with α>0 and one of the end points is variable say (x_f, y_f), i.e. (x_f, y_f) can move turning into ($x_f+\delta x_f$, $y_f+\delta y_f$), with prescribed condition on fixed boundary only $y(x_s)=y_s$ and

$$\delta y\big|_{x_s} = \delta y\big|_{x_k-s} = \delta y\big|_{x_k+s} = 0, (k = 1,...,n)$$

…(3.5)

$$\Delta v = \int_{x_s}^{x_f+\delta x_f} F(x, y+\delta y, y^{(\alpha)} + \delta y^{(\alpha)})dx - \int_{x_s}^{x_1} F\left(x, y, y^{(\alpha)}\right)dx$$

$$= \int_{x_f}^{x_f+\delta x_f} F(x, y+\delta y, y^{(\alpha)} + \delta y^{(\alpha)})dx + \int_{x_s}^{x_f}\left(F(x, y+\delta y, y^{(\alpha)} + \delta y^{(\alpha)}) - F\left(x, y, y^{(\alpha)}\right)\right)dx$$

…(3.6)

As F in the first integral of the right – hand side of equation (3.6) is continuous, it will be transformed with the aid of the mean value theorem to get:

$$\int_{x_f}^{x_f+\delta x_f} F(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)})dx = F\big|_{x=x_f+\theta\delta x_f}\,\delta x_f$$

where (0<θ<1). Furthermore, by virtue of continuity of the function F,

$$F\big|_{x=x_f+\theta\delta x_f} = F\left(x, y, y^{(\alpha)}\right)\Big|_{x=x_f} + \varepsilon_1$$

where; $\varepsilon_1 \to 0$ as $\delta x_f \to 0$ and $\delta y_f \to 0$

Consequently;

$$\int_{x_f}^{x_f+\delta x_f} F(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)})dx = F\left(x, y, y^{(\alpha)}\right)\Big|_{x=x_f}\delta x_f$$

…(3.7)

Since F is discontinuous in the second integral of the right-hand side of eq. (4.6), therefore the integral

$$\int_{x_s}^{x_f} F(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - F\left(x, y, y^{(\alpha)}\right)dx$$

can be transformed into

$$\int_{x_s}^{x_1-s}\left(F(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - F(x, y, y^{(\alpha)})\right)dx + \sum_{k=1}^{n-1}\int_{x_k+s}^{x_{k+1}-s}\left(F(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - F(x, y, y^{(\alpha)})\right)dx$$
$$+\int_{x_n+s}^{x_f}\left(F(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - F(x, y, y^{(\alpha)})\right)dx$$

and by using Taylor formula on each integrand to get:

$$= \int_{x_s}^{x_1-s}\left(F_y(x, y, y^{(\alpha)})\delta y + F_{y^{(\alpha)}}(x, y, y^{(\alpha)})\delta y^{(\alpha)}\right)dx + R\Big] +$$
$$\sum_{k=1}^{n-1}\int_{x_k+s}^{x_{k+1}-s}\left(F_y(x, y, y^{(\alpha)})\delta y + F_{y^{(\alpha)}}(x, y, y^{(\alpha)})\delta y^{(\alpha)}\right)dx + R\Big] +$$
$$+\int_{x_n+s}^{x_f}\left(F_y(x, y, y^{(\alpha)})\delta y + F_{y^{(\alpha)}}(x, y, y^{(\alpha)})\delta y^{(\alpha)}\right)dx + R\Big]$$

…(3.8)

(where R is an infinitesimal of higher order than δy or $\delta y^{(\alpha)}$).

Using (2.3), in the second term of all the integrand in (3.8), in which δy is α-differentiable, therefore

$$\int_{x_s}^{x_f}\left(F(x,y+\delta y,y^{(\alpha)}+\delta y^{(\alpha)})-F(x,y,y^{(\alpha)})\right)dx$$

$$=F_{y^{(\alpha)}}\delta y\Big]_{x_s}^{x_1}-\int_{x_s}^{x_1-s}\left(F_y-\frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx+\sum_{k=1}^{n-1}\left\{F_{y^{(\alpha)}}\delta y\Big]_{x_k+s}^{x_{k+1}}-\int_{x_k+s}^{x_{k+1}-s}\left(F_y-\frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx\right\}$$

$$+F_{y^{(\alpha)}}\delta y\Big]_{x_n+s}^{x_f}-\int_{x_n+s}^{x_f}\left(F_y-\frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\right)\delta y dx$$

Since the value of the functional are only along extremals, consequently

$$F_y-\frac{d^\alpha}{dx^\alpha}F_{y^{(\alpha)}}\equiv 0 \qquad \ldots(3.9)$$

and since the points (x_s, $y(x_s)$), ($x_k-\varepsilon$, $y(x_k-\varepsilon)$), ($x_k+\varepsilon$, $y(x_k+\varepsilon)$) are fixed (k=1,…,n), therefore

$$\int_{x_0}^{x_f}\left(F_y\delta y+F_{y^{(\alpha)}}\delta y^{(\alpha)}\right)dx\cong F_{y^{(\alpha)}}\delta y\big|_{x_f} \qquad \ldots(3.10)$$

Observe that $\delta y|_{x=x_f}$ does not mean the same as δy_f, the increment of y_f, for δy_f is the change of y-coordinate of the free end point, when it is moved from (x_f, y_f) to ($x_f+\delta x_f$, $y_f+\delta y_f$), whereas; $\delta y|_{x=x_f}$ is the change of y-coordinate of an extremal produced at the point $x=x_f$ when this extremal changes from one that passes through the points (x_s, y_s) and (x_f, y_f) to another one passing through (x_s, y_s) and ($x_f+\delta x_f$, $y_f+\delta y_f$), we can set (see [8], ch. 2)

$$\delta y\big|_{x=x_f}\cong\delta y_f-y'(x_f)\delta x_f$$

Consequently, from (3.7) and (3.10), and since the fundamental necessary condition for extremum $\delta v=0$, we have

$$\delta v=F\big|_{x=x_f}\delta x_f+F_{y^{(\alpha)}}\delta y\big|_{x=x_f}\equiv 0$$

$$=F\big|_{x=x_f}\delta x_f+F_{y^{(\alpha)}}(\delta y_f-y'\delta x_f)\equiv 0$$

$$=(F-y'F_{y^{(\alpha)}})\big|_{x=x_f}\delta x_f+F_{y^{(\alpha)}}\big|_{x=x_f}\delta y_f=0 \qquad \ldots(3.11)$$

If the variations δx_f and δy_f are independent, then we have the following condition for extremum:

$$F - y'F_{y^{(\alpha)}}\Big|_{x=x_f} \equiv 0$$

…(3.12a)

$$F_{y^{(\alpha)}}\Big|_{x=x_f} \equiv 0$$

…(3.12b)

If the variations δx_f and δy_f, are dependent, for instance, suppose the end point (x_f, y_f) can move along a certain curve $y_f=\varphi(x_f)$, we get

$$(F - y'F_{y^{(\alpha)}})\Big|_{x=x_f} \delta x_f + F_{y^{(\alpha)}}\Big|_{x=x_f} (\varphi'(x_f))\delta x_f \equiv 0$$

$$F - (y' - \varphi'(x))F_{y^{(\alpha)}}\Big|_{x=x_f} \delta x_f \equiv 0$$

Since δx_f is arbitrary, then the necessary condition which is called "transversality condition" becomes:

$$F - (y' - \varphi'(x))F_{y^{(\alpha)}}\Big|_{x=x_f} \equiv 0$$

…(3.13)

Now, we consider the functional of the form:

$$v\left(x, y(x), y^{(\alpha)}, y^{(\beta)}\right) = \int_{x_s}^{x_f} F\left(x, y, y^{(\alpha)}, y^{(\beta)}\right)dx$$

where F is discontinuous on (x_k), for k=1,…,n, with α, β>0 nonintegers and one of the end points is variable (say (x_f, y_f)), i.e. (x_f, y_f) can move turning into $(x_f+ \delta x_f , y_f+ \delta y_f)$, with prescribed condition on fixed boundary only, and $\delta y\big|_{x_s} = \delta y\big|_{x_k - s} = \delta y\big|_{x_k + s} = 0$ Now we proceed as in section (3.1.1), to get the following condition:

$$F_y - \frac{d^\alpha}{dx^\alpha} F_{y^{(\alpha)}} - \frac{d^\beta}{dx^\beta} F_{y^{(\beta)}} \equiv 0$$

…(3.14)

and since the point (x_s, y_s) is fixed, it follows that $\delta y\big|_{x_s} \equiv 0$, and therefore

$$\int_{x_0}^{x_1} \left(F_y \delta y + F_{y^{(\alpha)}} \delta y^{(\alpha)} + F_{y^{(\beta)}} \delta y^{(\beta)}\right)dx \cong F_{y^{(\alpha)}} \delta y\big|_{x=x_f} + F_{y^{(\beta)}} \delta y\big|_{x=x_f}$$

applying the same philosophy from Fig.1, we obtained the following conditions

$$\left(F - y'(F_{y^{(\alpha)}} + F_{y^{(\beta)}})\right)\Big|_{x=x_f} \delta x_f + (F_{y^{(\alpha)}} + F_{y^{(\beta)}})\Big|_{x=x_f} \delta y_f = 0$$

If the variations δx_f and δy_f are independent, then we have the following condition for extremum:

$$\left(F - y'(F_{y^{(\alpha)}} + F_{y^{(\beta)}})\right)\Big|_{x=x_f} = 0$$

…(3.15a)

$$(F_{y^{(\alpha)}} + F_{y^{(\beta)}})\Big|_{x=x_f} = 0$$

…(3.15b)

If the variations δx_f and δy_f are dependent, for instance, suppose the end point (x_f, y_f) can move along a certain curve $y_f=\varphi(x_f)$, we get

$$(F - y'(F_{y^{(\alpha)}} + F_{y^{(\beta)}}))\Big|_{x=x_f} \delta x_f + (F_{y^{(\alpha)}} + F_{y^{(\beta)}})\Big|_{x=x_f} \left(\varphi'(x_f)\right)\delta x_f \equiv 0$$

$$F - \left(y' - \varphi'(x)\right)\left(F_{y^{(\alpha)}} + F_{y^{(\beta)}}\right)\Big|_{x=x_f} \delta x_f \equiv 0$$

Since δx_1 is arbitrary, then the necessary condition becomes:

$$F - \left(y' - \varphi'(x)\right)\left(F_{y^{(\alpha)}} + F_{y^{(\beta)}}\right)\Big|_{x=x_f} \equiv 0$$

…(3.16)

Next, we are extended our results to different multi-fractional order $\alpha_i > 0$, $(i=1,2,\dots,m)$, of the following general problem:

$$v\left(x, y(x), y^{(\alpha)}, \dots, y^{(\alpha_m)}\right) = \int_{x_s}^{x_f} F\left(x, y, y^{(\alpha)}, \dots, y^{(\alpha_m)}\right) dx$$

…(3.17)

Applying the previous procedure, to obtain the following necessaries conditions:

$$F_y - \sum_{j=1}^{m} \frac{d^{\alpha_i}}{dx^{(\alpha_i)}} F_{y^{(\alpha_i)}} \equiv 0$$

…(3.18)

and since

$$\int_{x_s}^{x_f}\left(F_y\delta y-\sum_{j=1}^{m}\frac{d^{\alpha_i}}{dx^{(\alpha_i)}}F_{y^{(\alpha_i)}}\delta y^{(\alpha_i)}\right)dx\cong\left[\sum_{j=1}^{m}F_{y^{(\alpha_i)}}\delta y\right]_{x=x_f}$$

…(3.19)

and since

$$\delta y\big|_{x_f}\cong\delta y_f-y'(x_f)\delta x_f$$

Substitute it in (3.19), we obtain:

$$\delta v=F\big|_{x=x_f}\delta x_f+\left(\sum_{i=1}^{m}F_{y^{(\alpha_i)}}\Big|_{x_f}(\delta y_f-y'(x_f))\right)\delta x_f=0$$

…(3.20)

Rearranged it to get:

$$\delta v=F-(y'\sum_{i=1}^{m}F_{y^{(\alpha_i)}}\Big|_{x_f})\delta x_f+\sum_{i=1}^{m}F_{y^{(\alpha_i)}}\Big|_{x_f}\delta y_f=0$$

…(3.21)

If δx_f and δy_f are independent, then their coefficients should be vanish at $x=x_f$.

$$\left(F-(y'\sum_{j=1}^{m}F_{y^{(\alpha_i)}})\right)\Bigg|_{x_f}=0$$

…(3.22a)

$$\left(\sum_{j=1}^{m}F_{y^{(\alpha_i)}}\right)\Bigg|_{x_f}=0$$

…(3.22b)

which are the necessary conditions.

If δx_f and δy_f are dependent, then there is some relation between them, for instance, let $y_f=\varphi(x_f)$, we have

$$\delta v=F-(y'\sum_{i=1}^{m}F_{y^{(\alpha_i)}}\Big|_{x_f})\delta x_f+\sum_{i=1}^{m}F_{y^{(\alpha_i)}}\Big|_{x_f}\varphi'(x_f)\delta x_f=0$$

$$(F-(y'-\varphi'(x_f))\sum_{i=1}^{m}F_{y^{(\alpha_i)}})\Big|_{x_f}\delta x_f=0$$

$$(F-(y'-\varphi'(x_f))\sum_{i=1}^{m}F_{y^{(\alpha_i)}})\Big|_{x_f}=0$$

...(3.23)

which is the necessary condition.

Constrained Variational Problems

First, we are considering the functional of the form:

$v\left(x, y(x), y^{(\alpha)}\right)=\int_{x_s}^{x_f} F\left(x, y, y^{(\alpha)}\right)dx$, subject to $\phi\left(x, y(x), y^{(\alpha)}\right)=0$

…(3.24)

(Fixed Boundaries):

Our approach based on the theories presented in [6], and extended to our problems. Therefore, we construct the following auxiliary functional:

$Z(x, y(x), y^{\alpha}(x))=F+\lambda\phi$

…(3.25)

where λ is a lagrange multiplier, then the problem (3.24) can be stated as following

$$v\left(x, y(x), y^{(\alpha)}\right)=\int_{x_s}^{x_f} Z\, dx$$

…(3.26)

Since F is discontinuous on (x_k), for k=1,…,n, with α>0, and prescribed condition on fixed boundaries, and $\delta y\big|_{x_s}=\delta y\big|_{x_k-s}=\delta y\big|_{x_k+s}=\delta y\big|_{x_f}=0$, then Z also discontinuous on (x_k), and the fundamental lemma of the calculus of variation can't be applied, then it is more convenient to calculate the value of $v\left(x, y(x), y^{(\alpha)}\right)$ along the polygonal curves approximately, so it is convenient to replace the integral

$$\int_{x_s}^{x_f} Z\left(x, y, y^{(\alpha)}\right)dx \cong \int_{x_s}^{x_1-s} Z\left(x, y, y^{(\alpha)}\right)dx+\sum_{k=1}^{n-1}\int_{x_k+s}^{x_{k+1}-s} Z\left(x, y, y^{(\alpha)}\right)dx+\int_{x_n+s}^{x_f} Z\left(x, y, y^{(\alpha)}\right)dx$$

…(3.27)

Now by performing the classical steps as in [8] on each sub interval, for the extrema of (3.2), to obtain

$$\delta v = Z_{y^{(\alpha)}}\delta y\Big]_{x_s}^{x_1-s} - \int_{x_s}^{x_1-s}\left(Z_y - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx + \sum_{k=1}^{n-1}\left\{Z_{y^{(\alpha)}}\delta y\Big]_{x_k+s}^{x_{k+1}-s} - \int_{x_k+s}^{x_{k+1}-s}\left(Z_y - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx\right\}$$
$$+ Z_{y^{(\alpha)}}\delta y\Big]_{x_n+s}^{x_f} - \int_{x_n+s}^{x_f}\left(Z_y - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx = 0$$

The above formula can be written as follows

$$\delta v = Z_{y^{(\alpha)}}\delta y\Big]_{x_s}^{x_1-s} + \sum_{k=1}^{n-1}\left\{Z_{y^{(\alpha)}}\delta y\Big]_{x_k+s}^{x_{k+1}-s}\right\} + Z_{y^{(\alpha)}}\delta y\Big]_{x_n+1}^{x_f} - \int_{x_s}^{x_1-s}\left(Z_y - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx +$$
$$\sum_{k=1}^{n-1}\left\{\int_{x_k+s}^{x_{k+1}-s}\left(Z_y - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx\right\} + \int_{x_n+s}^{x_f}\left(Z_y - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx = 0$$

…(3.28)

Since, the first three terms are vanish, and then applying the fundamental lemma, on the integrand of the last three terms, we obtain the following condition

$\left(Z_y - \frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right) = 0$, which is equivalent to

$(F_y + \lambda\phi_y) - \frac{d}{dx^{(\alpha)}}(F_{y^{(\alpha)}} + \lambda\phi_{y^{(\alpha)}}) = 0$, with additional constraint $\phi(x, y, y^{(\alpha)}) = 0$ …(3.29)

Now, we are discussed the necessary conditions of the general form of the problem (3.24) including many dependent variables, higher integer and multi-fractional order derivatives

$\min\int_{x_s}^{x_f} F(x, y_1,..., y_n, y_1^{\alpha_1},..., y_1^{\alpha_m},..., y_n^{\alpha_1},..., y_n^{\alpha_m})dx$, with fractional order constraints

$\phi_j(x, y_1,..., y_n, y_1^{\alpha_1},..., y_1^{\alpha_m},..., y_n^{\alpha_1},..., y_n^{\alpha_m}) = 0$, j=1, …,I.

…(3.30)

and $y_l(x_s)$, $y_l(x_f)$ are known given, $\delta y|_{x_s} = \delta y|_{x_k-s} = \delta y|_{x_k+s} = \delta y|_{x_f} = 0$. (l=1,…,n & r=1,…,m)

By varieties one dependent variable say y_l and fixing the remaining dependent variables, then proceed as before. This procedure will be repeated for all dependent variables, we are getting the following necessary conditions

$\left[F_{y_l} + \sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l}\right] - \left[\sum_{r=1}^{m}\frac{d^{\alpha_r}}{dx^{\alpha_r}}(F_{y_l^{\alpha_r}} + \sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{\alpha_r}}\right] \equiv 0$, (l=1,…,n) as well as

$\phi_j\left(x, y_1,..., y_n, y_1^{\alpha_1},..., y_1^{\alpha_m},..., y_n^{\alpha_1},..., y_n^{\alpha_m}\right) = 0$, j=1, …,I.

…(3.31)

Moving Boundaries:

Finally, we are discussed the previous results **on moving boundaries**, by considering the following constrained variational problem:

$\min \int_{x_s}^{x_f} F\left(x, y_1,..., y_n, y_1^{\alpha_1},..., y_1^{\alpha_m},..., y_n^{\alpha_1},..., y_n^{\alpha_m}\right)dx$, with fractional order constraints

$\phi_j\left(x, y_1,..., y_n, y_1^{\alpha_1},..., y_1^{\alpha_m},..., y_n^{\alpha_1},..., y_n^{\alpha_m}\right) = 0$, j=1, …,I.

…(3.32)

$y_l(x_s)$ are known given, while x_f is movable.

Where F is discontinuous on (x_k), for k=1,…,n, with α>0 and one of the end points is variable say (x_f, y_f), i.e. (x_f, y_f) can move turning into ($x_f + \delta x_f$,$y_f + \delta y_f$), with prescribed condition on fixed boundary only $y(x_s)=y_s$ and $\delta y\big|_{x_s} = \delta y\big|_{x_k - s} = \delta y\big|_{x_k + s} = 0$. (l=1,…,n & r=1,…,m)

Using the same philosophies as before and the results in the previous sections, we are proceeding as in the following:

First, we consider the functional of the form:

$v\left(x, y(x), y^{(\alpha)}\right) = \int_{x_s}^{x_f} F\left(x, y, y^{(\alpha)}\right)dx$, subject to $\phi\left(x, y, y^{(\alpha)}\right) = 0$

…(3.33)

where; F is continuous, α>0 and one of the end points is variable (say (x_f, y_f)), i.e. (x_f, y_f) can move turning into ($x_f + \delta x_f$,$y_f + \delta y_f$), with prescribed conditions on fixed boundary only, and constructing the following auxiliary functional

$Z(x, y(x), y^{\alpha}(x)) = F + \lambda\phi$, where λ is a Lagrange multiplier. We proceed as following

$$\Delta v = \int_{x_s}^{x_f + \delta x_f} Z(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})dx - \int_{x_s}^{x_f} F(x, y, y^{(\alpha)})dx = \int_{x_f}^{x_f + \delta x_f} Z(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)})dx$$
$$- \int_{x_s}^{x_f} \left(Z(x, y + \delta y, y^{(\alpha)} + \delta y^{(\alpha)}) - Z(x, y, y^{(\alpha)})\right)dx$$

(3.34)

As F in the first integral of the right – hand side of equation (3.34) is continuous, it will be transformed with the aid of the mean value theorem to get:

$\int_{x_f}^{x_f+\delta x_f} Z(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)})dx = Z\big|_{x=x_f+\theta\delta x_f}\delta x_f$, where (0<θ<1). Furthermore, by virtue of continuity of the function F, $Z\big|_{x=x_f+\theta\delta x_f} = Z(x, y, y^{(\alpha)})\big|_{x=x_f} + \varepsilon_1$, where, $\varepsilon_1 \to 0$ as $\delta x_f \to 0$ and $\delta y_f \to 0$.

Consequently;

$$\int_{x_f}^{x_f+\delta x_f} Z(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)})dx = F + \lambda\phi\big|_{x=x_f}\delta x_f \quad \ldots(3.35)$$

Since F is discontinuous in the second integral of the right-hand side of eq. (3.34), therefore the integral

$$\int_{x_s}^{x_f} (Z(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - Z(x, y, y^{(\alpha)}))dx$$

can be transformed into

$$\int_{x_s}^{x_1-s}\left(Z(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - Z(x, y, y^{(\alpha)})\right)dx + \sum_{k=1}^{n-1}\int_{x_k+s}^{x_{k+1}-s}\left(Z(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - Z(x, y, y^{(\alpha)})\right)dx$$
$$+\int_{x_n+s}^{x_f}\left(Z(x, y+\delta y, y^{(\alpha)}+\delta y^{(\alpha)}) - Z(x, y, y^{(\alpha)})\right)dx$$

and by using Taylor formula on each integrand to get:

$$= \int_{x_s}^{x_1-s}\left(Z_y(x, y, y^{(\alpha)})\delta y + Z_{y^{(\alpha)}}(x, y, y^{(\alpha)})\delta y^{(\alpha)}\right)dx + R\Big] +$$
$$\sum_{k=1}^{n-1}\int_{x_k+s}^{x_{k+1}-s}\left(Z_y(x, y, y^{(\alpha)})\delta y + Z_{y^{(\alpha)}}(x, y, y^{(\alpha)})\delta y^{(\alpha)}\right)dx + R\Big] +$$
$$+\int_{x_n+s}^{x_f}\left(Z_y(x, y, y^{(\alpha)})\delta y + Z_{y^{(\alpha)}}(x, y, y^{(\alpha)})\delta y^{(\alpha)}\right)dx + R\Big] \quad \ldots(3.36)$$

(where R is an infinitesimal of higher order than δy or $\delta y^{(\alpha)}$).

Using (2.3), for in the second term of all the integrand in (3.36), in which ***δ***y is α-differentiable, therefore

$$\int_{x_s}^{x_f}\left(Z(x,y+\delta y,y^{(\alpha)}+\delta y^{(\alpha)})-Z(x,y,y^{(\alpha)})\right)dx$$

$$=Z_{y^{(\alpha)}}\delta y\Big]_{x_s}^{x_1-s}-\int_{x_s}^{x_1-s}\left(Z_y-\frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx+\sum_{k=1}^{n-1}\left\{Z_{y^{(\alpha)}}\delta y\Big]_{x_k+s}^{x_{k+1}-s}-\int_{x_k+s}^{x_{k+1}-s}\left(Z_y-\frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx\right\}$$

$$+Z_{y^{(\alpha)}}\delta y\Big]_{x_n+s}^{x_f}-\int_{x_n+s}^{x_f}\left(Z_y-\frac{d^\alpha}{dx^\alpha}Z_{y^{(\alpha)}}\right)\delta y dx$$

Since the value of the functional are only along extremals, consequently

$$(F+\lambda\phi)_y-\frac{d^\alpha}{dx^\alpha}(F+\lambda\phi)_{y^{(\alpha)}}\equiv 0$$

…(3.37)

and since the points(x_s, $y(x_s)$), ($x_k-\varepsilon$, $y(x_k-\varepsilon)$), ($x_k+\varepsilon$, $y(x_k+\varepsilon)$) are fixed (k=1,…,n), therefore

$$\int_{x_s}^{x_f}\left(Z_y\delta y+Z_{y^{(\alpha)}}\delta y^{(\alpha)}\right)dx\cong Z_{y^{(\alpha)}}\delta y\Big|_{x_f}$$

…(3.38)

Observe that $\delta y\big|_{x=x_f}$ does not mean the same as δy_f, as we mentioned it before, we have

$$\delta v=(F+\lambda\phi)_{x=x_f}\delta x_f+(F+\lambda\phi)_{y^{(\alpha)}}\delta y\big|_{x=x_f}\equiv 0$$

$$=(F+\lambda\phi)\big|_{x=x_f}\delta x_f+(F+\lambda\phi)_{y^{(\alpha)}}(\delta y_f-y'\delta x_f)\equiv 0$$

$$=((F+\lambda\phi)-y'(F+\lambda\phi)_{y^{(\alpha)}})\big|_{x=x_f}\delta x_f+(F+\lambda\phi)_{y^{(\alpha)}}\big|_{x=x_f}\delta y_f=0$$

…(3.39)

If the variations δx_f and δy_f are independent, then we have the following condition for extremum:

$$((F+\lambda\phi)-y'(F+\lambda\phi)_{y^{(\alpha)}})\big|_{x=x_f}\equiv 0$$

…(3.40a)

$$(F+\lambda\phi)_{y^{(\alpha)}}\big|_{x=x_f}\equiv 0$$

…(3.40b)

If the variations δx_f and δy_f, are dependent, for instance, suppose the end point (x_f, y_f) can move along a certain curve $y_f=\varphi(x_f)$, we get

$$(Z-y'Z_{y^{(\alpha)}})\big|_{x=x_f}\delta x_f+Z_{y^{(\alpha)}}\big|_{x=x_f}\left(\varphi'(x_f)\right)\delta x_f\equiv 0$$

$$Z-\left(y'-\varphi'(x)\right)Z_{y^{(\alpha)}}\Big|_{x=x_f}\delta x_f\equiv 0$$

Since δx_f is arbitrary, then the necessary condition which is called "transversality condition" becomes:

$$Z-\left(y'-\varphi'(x)\right)Z_{y^{(\alpha)}}\Big|_{x=x_f}\equiv 0$$

…(3.41)

Now, by variation one dependent variable say y_l and fixing the remaining dependent variables, then proceed as before. This procedure will be repeated for all dependent variables, we are getting the following necessary conditions

$$\left[F_{y_l}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l}\right]-\left[\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_{y_l^{\alpha_i}}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{\alpha_i}}\right]=0\text{, and}$$

$$(F+\sum_{j=1}^{I}\lambda_j\phi_j)-y_l'\left(\sum_{i=1}^{m}(F_{y_l^{(\alpha_i)}}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{(\alpha_i)}})\right)\Bigg|_{x_f}\delta x_1+\left(\sum_{i=1}^{m}(F_{y_l^{(\alpha_i)}}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{(\alpha_i)}})\right)\Bigg|_{x_f}\delta y_{l,f}=0,$$

(l=1,…,n).

…(3.42)

as well as $\phi_j\left(x,y_1,...,y_n,y_1^{\alpha_1},...,y_1^{\alpha_m},...,y_n^{\alpha_1},...,y_n^{\alpha_m}\right)=0$, (j=1,…,I)

If δx_f and δy_l are independent, then we have the following conditions:

$$\left[F_{y_l}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l}\right]-\left[\sum_{i=1}^{m}\frac{d^{\alpha_i}}{dx^{\alpha_i}}(F_{y_l^{\alpha_i}}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{\alpha_i}}\right]=0,$$

$$(F+\sum_{j=1}^{I}\lambda_j\phi_j)-y_l'\left(\sum_{i=1}^{m}(F_{y_l^{(\alpha_i)}}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{(\alpha_i)}})\right)\Bigg|_{x_f}=0,$$

$$\left(\sum_{i=1}^{m}(F_{y_l^{(\alpha_i)}}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{(\alpha_i)}})\right)\Bigg|_{x_f}=0\text{, (l=1,…,n).}$$

…(3.43)

as well as $\phi_j\left(x,y_1,...,y_n,y_1^{\alpha_1},...,y_1^{\alpha_m},...,y_n^{\alpha_1},...,y_n^{\alpha_m}\right)=0$, (j=1,…,I)

If δx_f and $\delta y_{l,f}$, are dependent, and suppose that (x_f, $y_{l,f}$) can move along a certain curve $y_{l,f}=\varphi_l(x_f)$, then we have

$$(F+\sum_{j=1}^{I}\lambda_j\phi_j)-(y_l'-\varphi_l')\left(\sum_{i=1}^{m}(F_{y_l^{(\alpha_i)}}+\sum_{j=1}^{I}\lambda_j(\phi_j)_{y_l^{(\alpha_i)}})\right)\Bigg|_{x_f}\delta x_f=0,\ (l=1,\dots,n).$$

as well as $\phi_j\left(x,y_1,...,y_n,y_1^{\alpha_1},...,y_1^{\alpha_m},...,y_n^{\alpha_1},...,y_n^{\alpha_m}\right)=0$, (j=1,…,I)

…(3.44)

Conclusion

The necessary conditions have developed for unconstrained and constrained fractional variational problems. The approach presented and the resulting equations are very similar to those for variational problems containing integer order only, the result of fractional calculus of variations reduce to those obtained from classical calculus of variations, in which, many of the concepts of the classical calculus of variations can be extended with minor modifications to fractional calculus of variations. Given the fact that many systems can be modeled more accurately using fractional derivative models, it is hoped that future research will continue in this area.

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شروط الأمثليه لمسائل التغاير ذات الرتب الكسرية

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المستخلص

في هذا البحث، تم أستخدام أسلوب تقريبي لمسائل التغاير غير المستمره، ذات المشتقات الكسريه، لأستنباط الشروط الضرورية لأمثلية مسائل التغاير المقيدة و الغير المقيدة التي تتضمن عدد من المشتقات ذات الرتب الصحيحة والكسرية ولعدد من المتغيرات المعتمدة (Dependent Variables) بالنسبة لمتغير مستقل Independent (Variable) واحد وعلى حدود ثابته ومتحركه.