CONSTRUCTION OF COMPLETE AND MAXIMAL (k, n) ARCS IN THE PROJECTIVE PLANE PG (2, 7)

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Abstract

The purpose of this paper is to study the construction of complete and maximal (k , n)-arcs in the projective plane PG (2 , 7) , n = 2 , 3, ...,8 .

A (k, n) –arc K in a projective plane is a set of K points such that no n + 1 of which are collinear. A (k, n) –arc is complete if it is not contained in a (k + 1, n) – arc.

A (k, n) – arc is a maximal if and only if every line in PG (2 , P) is a O – secant , or n – secant , which represented as (k , 2) – arc and (k , 8) – arc.

Introduction

Ahmad(1999) [4] studied the complete arcs in the projective plane over Galois Field GF(9) , also Rashad (1999) [10] showed the complete arcs in the projective plane over Galois Field GF(q) and Massa (2004) [8] studied the constriction of (k, n)- arcs from (k, m) – arcs in the PG(2, 17) for $2 \leq m < n$. Finally Najm (2005) [9] studied the constriction of (k, n)- arcs from (k, m) – arcs in the PG(2, 13) for $2 \leq m < n$. This paper deals with complete (k , n) – arc , maximal (k , n) – arc and how constructed from complete (k , m) – arc , $2 \leq m < n$.

The construction of complete (k , n) – arc , $n < k$ prepared from the union of some complete (k, m) – arc, $2 \leq m < n$. Usually the construction arc is incomplete arc and we get the complete by eliminating some points from the incomplete (k, n) – arc.

The only two maximal arcs are (k, 2) – arc and (k, 8) – arc which represented the whole plane since each line contains eight points.

Basic Definition

Definition (K , n) – Arcs [1, 2 , 6 , 7] : A (k , n) – arc in the projective plane PG (2 , P) is a set K points such that some line meets K in n points but no line meets k in more than n points $n \geq 2$, p is prime

Definition [4 ,6 , 9 , 10] : A (k , n) –arc is complete if it is not contained in**2.2**
(k + 1, n) – arc.

Definition [3 , 6 , 8 , 12] : A point p which is not on (k , n) – arc K has index i if there are exactly i (n – secant) through p , we dented the numbers of point p of index i by C_i .

Definition [5 , 6 , 9, 11] : A (k , n) – arc K is a maximal if and only if every line in PG (2 , p) is a O – secant or n – secant.

2.5 Definition PG(2, 7)[1, 6, 10]: A PG(2, 7) is the two – dimensional projective space which consists of points and lines with incidence relation between them and satisfying the following axioms:

i- Any two distinct lines are intersected in a unique point.

ii- Any two distinct points are contained in a unique line.

iii – There exist at least four points such that no three of them are collinear .

Remark (1)[4 , 5 , 6] : A (k , n) – arc K is complete if and only if C_0 = O ,we mean that C_0 is 0 (n – secant) , thus K is complete if and only if every point of PG (**2** , p) lies on some (n – secant) of K

The Projective Plane PG (2, 7)

The projective plane PG (2 , 7) contains 57 points and 57 lines , every line contains 8 points and every point is on 8 lines . Any line in PG (2 , 7) can be constructed by means of variety v. let Pi and Li , i = 1 , 2 , …,57 be the points and lines of PG(2 , 7) respectively. Let i stands for the points Pi and the lines Li, then all the points and the lines in PG (2, 7) are given in the table (1)

i	Pi	Li							
1	(1 , 0 ,0)	2	9	16	23	30	37	44	51
2	(0 , 1 , 0)	1	9	10	11	12	13	14	15
3	(1 , 1 , 0)	8	9	22	28	34	40	46	52
4	(2 , 1 , 0)	5	9	19	29	32	42	45	55
5	(3 , 1 , 0)	4	9	18	27	36	38	47	56
6	(4 , 1 , 0)	7	9	21	26	31	43	48	53
7	(5 , 1 , 0)	6	9	20	24	35	39	50	54
8	(6 , 1 , 0)	3	9	17	25	33	41	49	57
9	(0 , 0 , 1)	1	2	3	4	5	6	7	8
10	(1 , 0 , 1)	2	15	22	29	36	43	50	57
11	(2 , 0 , 1)	2	12	19	26	33	40	47	54
12	(3 , 0 , 1)	2	11	18	25	32	39	46	53
13	(4 , 0 , 1)	2	14	21	28	35	42	49	56
14	(5 , 0 , 1)	2	13	20	27	34	41	48	55
15	(6 , 0 , 1)	2	10	17	24	31	38	45	52
16	(0 , 1 , 1)	1	51	52	53	54	55	56	57
17	(1 , 1 , 1)	8	15	21	27	33	39	45	51
18	(2 , 1 , 1)	5	12	22	25	35	38	48	51

19	(3 , 1 , 1)	4	11	20	29	31	40	49	51
20	(4 , 1 , 1)	7	14	19	24	36	41	46	51
21	(5 , 1 , 1)	6	13	17	28	32	43	47	51
22	(6 , 1 , 1)	3	10	18	26	34	42	50	51
23	(0 ,2 , 1)	1	30	31	32	33	34	35	36
24	(1 , 2 , 1)	7	15	20	25	30	42	47	52
25	(2 , 2 , 1)	8	12	18	24	30	43	49	55
26	(3 , 2 , 1)	6	11	22	26	30	41	45	56
27	(4 , 2 , 1)	5	14	17	27	30	40	50	53
28	(5 , 2 , 1)	3	13	21	29	30	38	46	54
29	(6 ,2 , 1)	4	10	19	28	30	39	48	57
30	(0 , 3 , 1)	1	23	24	25	26	27	28	29
31	(1 , 3 , 1)	6	15	19	23	34	38	49	53
32	(2 ,3 ,1)	4	12	21	23	32	41	50	52
33	(3 ,3 ,1)	8	11	17	23	36	42	48	54
34	(4 , 3 , 1)	3	14	22	23	31	39	47	55
35	(5 ,3 , 1)	7	13	18	23	35	40	45	57
36	(6 , 3 ,1)	5	10	20	23	33	43	46	56
37	(0 ,4 ,1)	1	44	45	46	47	48	49	50
38	(1 , 4 ,1)	5	15	18	28	31	41	44	54
39	(2 , 4 ,1)	7	12	17	29	34	39	44	56
40	(3 ,4 ,1)	3	11	19	27	35	43	44	52
41	(4 ,4 ,1)	8	14	20	26	32	38	44	57
42	(5 ,4 ,1)	4	13	22	24	33	42	44	53
43	(6 , 4 ,1)	6	10	21	25	36	40	44	55
44	(0 , 5 ,1)	1	37	38	39	40	41	42	43
45	(1 ,5 ,1)	4	15	17	26	35	37	46	55
46	(2 ,5 ,1)	3	12	20	28	36	37	45	53
47	(3 ,5 ,1)	5	11	21	24	34	37	47	57

48	(4, 5, 1)	6	14	18	29	33	37	48	52
49	(5, 5, 1)	8	13	19	25	31	37	50	56
50	(6 ,5 ,1)	7	10	22	27	32	37	49	54
51	(0, 6 ,1)	1	16	17	18	19	20	21	22
52	(1, 6, 1)	3	15	16	24	32	40	48	56
53	(2, 6 ,1)	6	12	16	27	31	42	46	57
54	(3, 6, 1)	7	11	16	28	33	38	50	55
55	(4, 6, 1)	4	14	16	25	34	43	45	54
56	(5, 6 ,1)	5	13	16	26	36	39	49	52
57	(6, 6, 1)	8	10	16	29	35	41	47	53

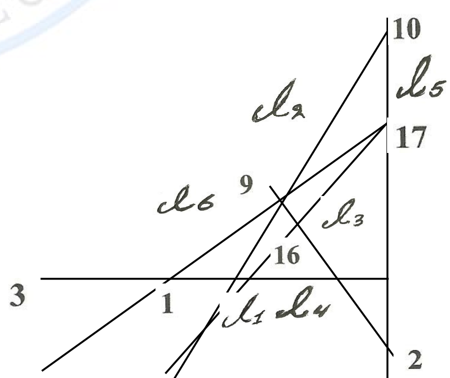
Table (1)

(Contains 57 points and 57 lines, every line contains 8 points and every point is on 8 lines)

4- The Construction of (k, n) – Arcs in PG (2, 7) :

Let A = { 1,2,9,17} be the set reference and unit points in the table (1) such that 1 = (1 , 0 , 0), 2 = (0 , 1 , 0), 9 = (0 , 0 , 1), 17 = (1 , 1 , 1) . A is a (4 , 2) – arc , since no three points of A are collinear , the points of A are the vertices of a quadrangle whose side are the lines

ℓ_1 = [1,2] = { 1,2,3,4,5,6,7,8}
ℓ_2 = [1,9] = { 1,9,10,11,12,13,14,15 }
ℓ_3 = [1,17] = { 1,16,17,18,19,20,21,22 }
ℓ_4 = [2,9] = { 2,9,16,23,30,37,44,51 }
ℓ_5 = [2,17] = { 2,10,17,24,31,38,45,52 }
ℓ_6 = [9,17] = { 3,9,17,25,33,41,49,57 }



The diagonal points of A are the points {3, 10, 16} where:

3 = ∩

10 = $\ell_2 \cap \ell_5$;

16 = $\ell_3 \cap \ell_4$, which are the intersection points of pairs of the opposite sides. Then there are 37 points on the sides of the quadrangle , four of them are points of the arc A , and

three of them are diagonal points of A , so there are 20 points not on the sides of the quadrangle which are the points of index zero for A these points are:{24,27,28,29,32,34,35,36,39,40,42,43,46,47,48,50,53,54,55,56 }

Hence A is incomplete (4, 2) – arc

The Conics In PG (2, 7) Through the Reference and Unit points The general equation of conic is

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0 \quad \ldots\ldots (1)$$

By substituting the points of the arc – A in (1) , we get

$1 = (1, 0, 0) \rightarrow a_1 = 0$

$2 = (0, 1, 0) \rightarrow a_2 = 0$

$9 = (0, 0, 1) \rightarrow a_3 = 0$

$17 = (1, 1, 1) \rightarrow a_4 + a_5 + a_6 = 0$

So equation (1) becomes

$$a_4 x_1 x_2 + a_5 x_1 x_3 + a_6 x_2 x_3 = 0 \quad \ldots\ldots (2)$$

If $a_4 = 0$, then $a_5 x_1 x_3 + a_6 x_2 x_3 = 0$

Hence $x_3 (a_5 x_1 + a_6 x_2) = 0$, $x_3 = 0$ or $a_5 x_1 + a_6 x_2 = 0$

Which are a pair of lines, then the conic is degenerated, therefore $a_4 \neq 0$

Similarly $a_5 \neq 0$ and $a_6 \neq 0$

Dividing equation (2) by a_4 we get

$$x_1 x_2 + \frac{a_5}{a_4} x_1 x_3 + \frac{a_6}{a_4} x_2 x_3 = 0$$

$$x_1 x_2 + \alpha x_1 x_3 + \beta x_2 x_3 = 0 \quad \ldots\ldots (3)$$

$\alpha = \dfrac{a_5}{a_4}$, $\beta = \dfrac{a_6}{a_4}$, then

$1 + \alpha + \beta = 0 \pmod{7}$

$\beta = -(1 + \alpha) + Z K$, then (3) can be written as:

$x_1 x_2 + \alpha x_1 x_3 - (1 + \alpha) x_2 x_3 = 0$ ……(4)

Where $\alpha \neq 0$ and $\alpha \neq 6$, for if $\alpha = 0$ or $\alpha = 6$,we get degenerated conic , i.e α = 1 ,2 ,3 ,4 ,5

The Equations and the Points of the Conic of PG (2, 7)Through The Reference and Unit Points

For any value for α there is a unique conic containing eight points, four of them are the reference and unit points

1- If $\alpha = 1$, then the equation of the conic C_1 is

$x_1 x_2 + x_1 x_3 + 5 x_2 x_3 = 0$, the point of C_1 are { 1,2,9,17,29,35,40,48,}

which is a complete (7 , 2) – arc , since there are no points of index zero for C_1

2- If $\alpha = 2$, then the equation of the conic C2 is

$x_1 x_2 + 2 x_1 x_3 + 4 x_2 x_3 = 0$

The points of C2 are {1,2,9,17,28,36,39,55} , which is a complete (7, 2) –arc, since there are no points of index zero for C_2

3- If $\alpha = 3$, then the equation of the conic C_3 is

$x_1 x_2 + 3x_1 x_3 + 3x_2 x_3 = 0$

The points of C_3 are {1,2,9,17,26,32,50,56} , which is a complete (7, 2) – arc, since there are no points of index zero for C_3

4- If $\alpha = 4$, then the equation of the conic C_4 is

$x_1 x_2 + 4x_1 x_3 + 2 x_2 x_3 = 0$

The points of C_4 are {1,2,9,17,27,43,46,54 } , which is a complete (7, 2) – arc since there are no points of index zero for C_4

5- If $\alpha = 5$, then the equation of the conic C_5 is

$x_1 x_2 + 5 x_1 x_3 + x_2 x_3 = 0$

The points of C_5 are {1,2,9,17,34,42,47,53} , which is complete (7 , 2) – arc , since there are no points of index zero for C_5

Thus there are five complete (7 , 2) – arcs in the PG (2 , 7) which are

C_1 = {1,2,9,17,29,35,40,48}

C_2 = {1,2,9,17,28,36,39,55}

C_3 = {1,2,9,17,26,32,50,56}

C_4 = {1,2,9,17,27,43,46,54}

C_5 = {1,2,9,17,34,42,47,53}

Construction of Complete (k, 3) –Arcs

We get complete (k, 3) – arcs through the following steps:

We take the union of two complete (8, 2) – arcs, say C_1 and C_2 denoted by D_1.

a . Let $D_1 = C_1 \cup C_2$ = { 1,2,9,17,28,29,35,36,39,40,48,55 }, we notice that D_1 is incomplete (k , 3) – arc , since there exist the points {3 ,5,16,18,45,47,51,53} of index zero for D_1

B – We add the point {3} from the index zero to D_1, therefore

D_1^1 = {1,2,3,9,17,28,29,35,36,39,40,48,55} is a complete (13 , 3) – arc , since there is no point of index zero i.e $C_0 = 0$.

Let $D_2 = C_1 \cup C_3$ = { 1,2,9,17,26,29,32,35,40,48,55 } , we notice that there are some line meet D_2 in four points , hence (k , 3) is not complete. So we eliminate some points from D_2 to determine a complete (k , 3) – arc as follows:

Let $D_2^1 = C_1 \cup C_3$ / {48} = { 1,2,9,17,26,29,32,35,40,50,56} , we notice that D_2 is incomplete (k, 3) – arc ,since there exist the points of index zero for D_2 which are { 8,10,11,13,18,38,41,51,52 }

We add {8, 11} from the index zero to D_2 , therefore

D_2 = {1,2,8,9,11,17,26,29,32,35,40,50,56} is a complete (13 , 3) – arc , since C_0 =0

Let $D_3 = C_1 \cup C_4$ = {1,2,9,17,27,29,35,40,43,46,48,54 } , notice that D_3 is incomplete (k, 3) – arc, since there exist points of index zero for D_3 which are {10,16,18,32,51,56} we add { 56 } from the index zero to D_3 , therefore

D_3^1 = { 1,2,9,17,27,29,35,40,43,46,48,54,56 } is a complete (13 , 3) –arc ,since C_0 =0

Let $D_4 = C_1 \cup C_5$ = { 1,2,9,17,29,34,35,40,42,47,48,53 } , notice that there are some line meet D_4 in four point , hence (k , 3) is not complete. So we eliminate some points from D_4 to determine a complete (k , 3) – arc as follows

Let $D_4 = C_1 \cup C_5$ / {53} = {1,2,9,17,29,34,35,40,42,47,48 } , we notice that D_4 is incomplete (k, 3) –arc since there exist points of index zero for D_4 which are {3,4,15,18,24,25,51,57}

We add {4, 25} from the index zero to D_4 , therefore

D_4^1 = {1,2,4,9,17,25,29,34,35,40,42,47,48} is a complete (13 , 3) – arc since $C_0 = 0$.

Construction of Complete (k, 4) – Arcs

Let $E_1 = D_1 \cup D_2$ = {1,2,3,8,9,11,17,26,28,29,32,35,36,39,40,48,50,56 } , we notice that there are some line meet E_1 in five points , and hence E_1 is not complete (k , 4) – arc there for we eliminate {32 , 48} from it to determine complete (k , 4) – arc as follows ,

$E_1 = D_1^1 \cup D_2^1$ / {32 , 48 } = {1,2,3,8,9,11,17,26,28,29,35,36,39,40,50,56}, E_1 is incomplete since there are points of index zero which are {5,10,13,18,19,30,33,41,45,47,51,53} .

We add {5, 10, 31} from the index zero to E_1, therefore

E_1^1 = {1,2,3,5,8,9,10,11,17,26,28,29,31,35,36,39,40,50,56 },is a complete (19, 4) – arc, since $C_0 = 0$.

Let $E_2 = D_1^1 \cup D_3^1$ = {1,2,3,9,17,27,28,29,35,36,39,40,43,46,48,54,55,56 } , notice that E is not complete (k , 4) – arc , since there are points of index zero which are { 5 , 10 , 31 ,33 , 45 }

We add {5, 31} from index zero to E_2 , therefore

E_2^1 = {1,2,3,5,9,17,27,29,31,35,36,39,40,43,46,48,54,55,56 } is a complete (19 – 4) – arc, since there is no point of index zero.

Let $E_3 = D_1^1 \cup D_4^1$ = {1,2,3,4,9,17,25,28,29,34,35,36,39,40,42,47,48,55} . Notice that E_3 is not complete (k , 4) –arc , since there are points of index zero which are {16 , 50 , 51 , 53}

We add {16, 50} from index zero to E_3 , therefore

E_3^1 = { 1,2,3,4,9,16,17,25,28,29,34,35,36,39,40,42,47,48,50,55 } is complete (20, 4) – arc, since there is no points of index zero.

Construction of Complete (k , 5) – Arcs

Let $F_1 = E_1^1$ U E_2^1 = {1,2,3,5,8,9,10,11,17,26,27,28,29,31,35,36,39,40,43,46,48,50,54,55,56 } , notice that there is a line meet F_1 in six points , hence (k , 5) is not complete. So we eliminate a point {11} from F_1 to determine a complete (k , 5) – arc as follows

Let $F_1 = E_1$ U E_2 /{ 11 }= {1,2,3,5,8,9,10,17,26,27,28,29,31,35,36,39,40,43,46,48,50,54,55,56 }. Notice that F_1 is incomplete since there exist points of index zero which are {12,13,16,18,19,32,38,41,44,45,47,49,51}

We add {12 , 13, 18, 32} from index zero to F_1 , then

F_1^1 = {1,2,3,5,8,9,10,12,13,17,18,26,27,28,29,31,32,35,36,39,40,43,46,48,50,54,55,56} is a complete (28 , 5) – arc , since $C_0 = 0$

Let $F_2 = E_1$ U E_3 = {1,2,3,4,5,8,9,10,11,16,17,25,26,28,29,31,34,35,36,39,40,42,47,48,50,55,56}

Notice that there are some lines meet F_2 in six points , hence (k , 5) is un complete. So we eliminate some points from F_2 to determine a complete(k , 5) – arc as follows

Let $F_2 = E_1^1$ U E_3^1 / {3, 16, 17} = { 1,2,4,5,8,9,10,11,25,26,28,29,31,34,35,36,39,40,42,47,48,50,55,56 } , we notice that F_2 is incomplete, since there exist the points of index zero which are { 12,15,43 }

We add {15} from index zero to F_2 , then

F_2^1 ={ 1,2,4,5,8,9,10,11,15,25,26,28,29,31,34,35,36,39,40,42,47,48,50,55,56 } is a complete (25, 5) – arc, since $C_0 = 0$.

Construction of Complete (k, 6) – Arcs

Let $G = F_1^1$ U F_2^1 = {1,2,3,4,5,8,9,10,11,12,13,15,17,18,25,26,27,28,29,31,32,34,35,36,39,40,42,43,46,47,48,50,5

4,55,56 } , notice that there are some lines meet G in seven points ,hence (k , 6) – arc is incomplete arc. So we eliminate some points from G to determine a complete (k , 5) – arc as follows.

Let G = F_1^1 U F_2^1 / {9, 39, 42} =
{1,2,3,4,5,8,10,11,12,13,15,17,18,25,26,27,28,29,31,32,34,35,36,40,43,46,47,48,50,54,55,56 } .

Notice that G is incomplete, since there exist the points of index zero for G which are {21,45,49,52} . We add {21,49} from index zero to G , therefore
G^1={1,2,3,4,5,8,10,11,12,13,15,17,18,21,25,26,27,28,29,31,32,34,35,36,40,43,46,47,48,49,50 ,54,55,56} is a complete (34 , 6) – arc , since C_0 = 0.

Construction of Complete (k, 7) – Arcs

Let us take complete (k , 6) – arc G^1 , G^1 is incomplete (k , 7) – arc , since there exist points of index zero for G^1 which are
{6,7,9,14,16,19,20,22,23,24,30,33,37,38,39,41,42,44,45,51,52,53,57}

We add eight points of index zero which are {9,16,20,33,39,42,44,57} to G^1 ,we denoted it by H
H={1,2,3,4,5,8,9,10,11,12,13,15,16,17,18,20,21,25,26,27,28,29,31,32,33,34,35,36,39,40,42,43,44,46,47,48,49,50,54,55,56,57 } is a complete (42 , 7) – arc , since C_0 = 0.

Construction of Complete (k, 8) – Arcs

We take complete (2 , 7) – arc H , H is incomplete (k , 8) – arc since there exist points of index zero for H which are {6,7,14,19,22,23,24,30,37,38,41,45,51,52,53 }

We add the points of index zero to H denoted by I , then I contains all the points of the plane i.e I = {1,2,3,…,55,56,57 } is a complete (57 , 8) – arc.

This arc is the whole plane, since each line in it contains eight points. Hence this arc is a maximal arc.

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بناء أقواس كاملة وأعظمية - (k , n) في المستوى الإسقاطي PG(7 ,2)

الخلاصة

الغرض من هذا البحث هو دراسة بناء أقواس كاملة واعظمية - (k , n) n = 1,2,…,8 في المستوى الاسقاطي PG (2, 7).

قوس – (k , n) في المستوى الاسقاطي هو مجموعة من k من النقاط بحيث لا يوجد n + 1 نقطة منها على استقامة واحدة . قوس - (k , n) يكون كاملا أذا لم يكن محتوى في القوس - (k + 1 , n). قوس - (k ,n) يكون أعظم قوس أذا وفقط أذا كان كل مستقيم في (PG (2 ,p لا يقطع القوس أو يقطعه في n من النقاط ، وهي هنا تمثل القوسين (k , 2) و (k , 8).