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DIRECT AND INVERSE PROBLEMS FOR THREE- DIMENSIONAL INERIOR ACOUSTIC WAVE SCATTERING

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Abstract

The first aim of this research to solve the direct interior problem in a given region D in three space R^3 of a non-homogeneous and homogeneous Helmholtz's equation with boundary conditions. Secondly, using the Direct Variation method which is used successfully to solve the inverse problem for determining a form of region in space R^3 over which the Helmholtz's equation is defined or main about the solution of this equation is available in space R^3 Numerical simulation for this case is utilized. Some of the results tabulated, indicated that the direct variation method is reliable in solving mathematical inverse problems.

المستخلص

الهدف الأول في هذا البحث هو حل المسالة المباشرة لمعادلة Helmholtz غير المتجانسة وعلى منطقة داخلية في الفضاء الثلاثي R^3. والثاني استخدام طرق التغاير المباشرة لحل المسالة العكسية لايجاد شكل المنطقة المعرفة عليها معادلة Helmholtz عندما تكون بعض معلومات حل المعادلة متوفرة في منطقة ما في الفضاء الثلاثي R^3. وتمت مناقشة هذه المسالة عدديا وجدولت بعض النتائج التي توضح ان طرق التغاير تكون مناسبة جدا.

DIRECT AND INVERSE PROBLEMS FOR THREE- DIMENSIONAL INERIOR ACOUSTIC WAVE SCATTERING

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Introduction

The direct interior problem which is concerned here to determine the solution of the non-homogeneous Helmholtz equation in a region D in space R^3 with homogeneous boundary conditions. This problem can be solved by using the variation method (Ritz Method) as an inverse problem of calculus of variation [3, 4].

While the inverse interior problem that has also been considered here is to determine the shape of a region D in space R^3 in which the non-homogeneous Helmholtz equations is defined when the values of the solution of this equation are given at a finite number of a given points which lie in the unknown region and some of these points lie on its inverse problem is solved by using the direct variation method which is first used to solve an inverse eigenvalue problem, [5] and secondly used to solve the inverse acoustic scattering problem to determine a region D in space R^3 [6].In this method the problem of determining the unknown region is transformed to determine unknown parameters of the unknown region. Such parameters are determined by using a discrete form at least square approximation which named by representing a nonlinear optimization problem, the constrained Hook and Jeeves method becomes efficient in solving that problem, [1,7].

The direct interior problem for Helmholtz equation with its solution

The Statement of the problem:-

Let the region D be defined in three-dimensions as:

$$D=\{(x,y,z):-a \le x \le a, -h(x) \le y \le h(x), -g(x,y) \le z \le g(x,y),\}.$$

The non-homogeneous Helmholtz equation in three-dimensions is;

$$(\nabla^2 + k^2)u = f \quad on\ region\ D \tag{1}$$

where $u: D \to \mathcal{R}, u \in C^2(D) and\ f: D \to \mathcal{R}, f \in C\ \ (D).$

L is a linear operator defined by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2}$$

∇^2 is a Laplacian in three dimensional coordinate in the interior region ***D*** k= wave number, where k^2 is not an eigenvalue of the associated homogeneous problem and k^2∈R,D is a bounded regular domain in the three space (R3).

Assume that the bounded condition associated with equation (1) is

$$u = 0 \text{ on } \partial D . \tag{3}$$

The direct problem is to find the solution of equations (1) and (3) in the region D.

<u>Variational Formulation of the problem:-</u>

Before solving equation (1) &(3) in the region D, it is very important to mention that problem has a unique solution, since k^2 is not an eigenvalue for the associated homogeneous equation [4].

For solving this problem using the variational method (Ritz Method), it must be formulated as a variational problem. To make such formulation, first let $< u, v >$ be the symmetric, non degenerate, classical, bilinear form defined by:

$$< u, v >= \iiint_D uv dz dy dx , \tag{4}$$

where $u : D \to \mathcal{R}$ and $v : D \to \mathcal{R}$.

The linear operator L in (2) is symmetric with respect to the non-degenerate classical bilinear form (4), that is to say:

$$< Lu_1, u_2 > = < Lu_2, u_1 >, \qquad (5)$$

Where $u_1 = u_1(x, y, z)$ *and* $u_2 = u_2(x, y, z)$, both satisfying the boundary condition (3), and both belong to the domain of definition of L. Consequently, the critical points of the functional

$$\mathrm{J[u]} = \frac{1}{2} < Lu, u > = < f, u > \qquad (6)$$

are solutions of the boundary value equations [1,3] and [4].

Equation (6) can be formed as:

$$\mathrm{J[u]} = \frac{1}{2} \iiint_D (u_{xx} + u_{yy} + u_{zz} + k^2 u^2) dzdydx - \iiint_D fu \, dzdydx, \qquad (7)$$

Upon using divergent theorem, and vanishing of u on the boundary, equation (7) becomes

$$\mathrm{J[u]} = \frac{1}{2} \iiint_D (u_x^2 + u_y^2 + u_z^2 - k^2 u^2 + 2fu) dzdydx, \qquad (8)$$

Solution of the variational problem:-

To find the solution of u which is the critical point of equation (8), the Ritz method will be used[3]. The procedure utilized here, can be described by using the following steps:

Step 1: The solution of u is approximated by a linear combination of elements of a complete sequence of functions $\{Q(x, y, z)\}$ defined over the region D and each function vanishes on ∂D. In other words,

$$u_N(x, y, z) = \sum_{n=1}^{N} a_n \, Q_n(x, y, z) \qquad (9)$$

where $a_1, a_2, a_3, \dots, a_N$ coefficients to be determined. These functions can be defined as follows

$$Q_1 = (z^2, g^2(x,y),\ Q_2 = xQ_1,\ Q_3 = yQ_1, Q_4 = zQ_1, Q_5 = x^2Q_1,$$
$$,Q_6 = xyQ_1, , Q_7 = xzQ_1\ ,Q_8 = y^2Q_1, , ,Q_9 = yzQ_1, ,Q_{10} = z^2Q_1, \quad \text{where } Q_i = Q_i(x,y,z)$$
$$\forall i = 1,2,\dots,10. \tag{10}$$

It should be mentioned that the above procedure can used to define other elements of the sequence of functions defined over the region D, and each of these functions vanishes along the boundary D. However, in this work N is taken to be in equation (10).

Step 2: Substituting u as given by expression (9) and its first partial derivatives with respect to x,y and z for their corresponding terms of u in equation (8), one obtains the following function for the unknown parameters a_n:

$$J[u] = -\frac{1}{2}\iiint_D \left(\left(\sum_{n=1}^{N} a_n \frac{\partial Q_a}{\partial x}\right)^2 + \left(\sum_{n=1}^{N} a_n \frac{\partial Q_a}{\partial y}\right)^2 + \left(\sum_{n=1}^{N} a_n \frac{\partial Q_a}{\partial z}\right)^2 - k^2\left(\sum_{n=1}^{N} a_n\, Q_a\right)^2 + 2f\sum_{n=1}^{N} a_n\, Q_a\right) dzdydx. \tag{11}$$

where $\bar{a} = (a_1, a_2, a_3, \dots a_N), Q_a = Q_n(x,y,z)\ \ and\ f = f(x,y,z).$

Step 3: To find the critical points of equations (10), one must determine the unknown parameters such that at such values equation (10) has a minimum value. The problem is thus reduced to find the minimum of function (10) with respect to $\bar{a}$, such problem is a non linear optimization problem. In this work it is solved by using unconstrained Hook and Jeeves method,

Step 4: Substitute the resulting values for $a_1, a_2, a_3, \dots a_N$ from the last step in equation(9) to obtain the approximate solution $u_n(x,y,z)$ of equations (1)and(3) in the region D.

Numerical Example:

Consider the direct interior problem for Helmholtz equation that have been discussed before, let a Helmholtz equation is given by:

$$u_{xx} + u_{yy} + u_{zz} + k^2 u^2 = f(x, y, z) \quad on\ region\ D$$

where

$$f(x,y,z) = 2\left(c^2 - \frac{c^2}{a^2}x^2 - \frac{c^2}{b^2}y^2\right)\left(\frac{2c^2b^2 + 2c^2a^2}{a^2b^2}\right) + 2\left(1 - \frac{2c^2}{a^2}x^2 - \frac{c^2}{b^2}y^2\right) + k^2\left(z + c^2 - \frac{c^2}{a^2}x^2 - \frac{c^2}{b^2}y^2\right)\left(z - c^2 - \frac{c^2}{a^2}x^2 - \frac{c^2}{b^2}y^2\right) \quad (12)$$

$a = 1; b = 1.5; c = 2\ and\ k = 2.1$.

associated with the boundary condition,

$$u = 0 \ \ on\ \partial D.$$

The region D in section 1.1 is used, where,

$$h(x) = \sqrt{b^2 - \left(\frac{bx}{a}\right)^2} \quad and\ g(x, y) = c^2\ \left(1 - \frac{1}{a^2}x^2 - \frac{1}{b^2}y^2\right) \quad (13)$$

A computer program is written and is used to solve the direct interior problem for Helmholtz equation which is considered above. The problem is solved by using the unconstrained Hook and Jeeves method optimization method to minimize function (10) with respect to the unknowns $a_n (n = 1,2,3, \dots, N)$ which are the coefficients of the approximate solution (9) where N=10 are tabulated below:

Table (1), The results of solution by computer program for Helmholtz's equation for different value of a,b and c. b and c

n	x_n
1	0.99927
2	0
3	-1.94289×10^{-16}
4	1.94289×10^{-16}
5	7.89×10^{4}
6	1.99429×10^{-16}
7	1.94289×10^{-15}
8	-3.89999×10^{-4}
9	0
10	8.39999×10^{-4}

The iInverse interior problem for Helmholtz equation in $\mathcal{R}^3$

Description of the problem:-

The previous part (section 1) us devoted to the solution of the interior direct problem for Helmholtz equation in space $\mathcal{R}^3$ in which the solution of the equation is found over the interior of a given region. Determination of the inverse of this problem is the region D when the solution of Helmholtz equation is given over a subset Γ (at discrete points or otherwise) of the region D. In both cases (direct and inverse) it is assumed that the solution vanishes over the boundary (known or unknown) of region D.

Before indulging in the details of the steps for finding the solution too worthwhile to mention that the unknown region may not be cubic; hence it is assumed that the boundary ∂D of the region D can be expressed by:

$$\partial \mathrm{D} = \begin{cases} \left.\begin{aligned} y = a_1(x) \; where \; y < 0 \\ y = a_2(x) \; where \; y > 0 \end{aligned}\right\} for \; -a < x < a \\ \left.\begin{aligned} z = b_1(x,y) \; where \; z < 0 \\ z = b_2(x,y) \; where \; z > 0 \end{aligned}\right\} for \; -a_1(x) < y < a_2(x) \; and \; -a < x < a \end{cases} \tag{14}$$

Mathematical Statement of the Inverse Interior Problem of Helmholtz Equation:-

Before using the direct variational method, the boundary of the unknown region must be approximated by functions with unknown coefficients.

From the definition of ∂D as is given in equation (14) it seems that to determine the region D it suffices to determine the unknown interval (-a,a) and the continuous functions:

$$\left.\begin{aligned} h_1: (-a,a) \longrightarrow R^-, h_2: (-a,a) \longrightarrow R^+, \\ g_1: (-a,a) \times (-b,b) \longrightarrow R^-, g_2: (-a,a) \times (-b,b) \longrightarrow R^+ \end{aligned}\right\} \tag{15}$$

where b$\in R$

such that:

$$h_n(-a) = h_n(a) = 0 \;\; for \;\; n = 1,2,$$

and

$$g_n(-a,0) = g_n(a,0) = g_n(0,-b) = g_n(0,b) = 0 \quad for \;\; n = 1,2,$$

Now, assume that:

$$\left.\begin{aligned} h_1(x,\bar{\alpha}) = \sum_{r=0}^{L} \alpha_r x_r, h_2(x,\bar{\beta}) = \sum_{r=0}^{L} \beta_r x_r \;, \\ g_1(x,y,\bar{\gamma}) = \in \sum_{t=0}^{N}\sum_{s=0}^{M} \gamma_{st} x^s y^t \;\; and \; g_2(x,y,\bar{\eta}) = \sum_{t=0}^{N}\sum_{s=0}^{M} \eta_{st} x^s y^t \end{aligned}\right\} \tag{16}$$

For all x$\in$(-a,a),y$\in$(-b,b) where $\bar{\alpha} = (\alpha_0 \alpha_1 \dots, \alpha_L)$, $\bar{\beta} = (\beta_0 \beta_1 \dots, \beta_L)$,

$\bar{\gamma} = (\gamma_{00}, \gamma_{01}, \dots, \gamma_{m0}, \gamma_{01}, \gamma_{11}, \dots, \gamma_{M1}, \dots, \gamma_{0N}, \gamma_{1N}, \gamma_{2N}, \dots, \gamma_{MN})$

$\bar{\eta} = (\eta_{00}, \eta_{01}, \dots, \eta_{m0}, \eta_{01}, \eta_{11}, \dots, \eta_{M1}, \dots, \eta_{0N}, \eta_{1N}, \eta_{2N}, \dots, \gamma_{\eta MN})$

And $\alpha_r \beta_r \gamma_{st} \eta_{st}$ are unknown real numbers (for r=0,1,…,L;s=0,1,…,M; t=0,1,…,N) which will be determined.

The unknown region can be approximated by the region as well as

$$D(a, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta}) = \{(x, y, z) | x \in (-a, a), y \in (h_1(x, \bar{\alpha}), h_2(x, \bar{\alpha})), z \in (g_1(x, y, \bar{\eta}), g_1(x, y, \bar{\eta}))$$

To make the analysis easier it is assumed that:

i-The unknown region is convex.

ii-The unknown region in xy-plane is symmetric about the both x-axis and y-axis.

iii-The unknown region is symmetric about xy-plane.

If the unknown region is convex, it is advisable to assume that the approximate region $D(a, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$ must likewise be convex. This is insured if one assumes that the polynomials approximating the functions $h_1(x, \bar{\alpha})$ and $h_2(x, \bar{\beta})$ in the definition of the unknown region D must at most of order 2, that is L=2. From conditions (15.a) and assumption(ii) one can be obtained that the functions h_1 and h_2 can be defined in the form

$$h_1(x, a, b) = -\sqrt{b^2 - \frac{a^2}{b^2} x^2} \text{ and } h_2(x, a, b) = -\sqrt{b^2 - \frac{a^2}{b^2} x^2} \quad (17)$$

where a and b are unknown real numbers, to be found. From conditions (15.b) and assumption (iii), it follows that the functions h_1 and h_2 can be defined in the form

$$g_2(x, y, a, b, c) = c^2(1 - x^2/a^2 - y^2/b^2) \text{ and } g_1(x, y, a, b, c) = -g_2(x, y, a, b, c) \quad (18)$$

where c is unknown real number to be determined of the unknown region $D(a, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\eta})$ becomes in the form

$$D(Z) = \{(x, y, z) | x \in (-a, a), y \in \left(h_1(x, a, b), h_2(x, a, b)\right), z \in \left(g_1(x, y, a, b, c), g_1(x, y, a, b, c)\right)\ x, y, z \in \mathcal{R}\} \tag{19}$$

where $\bar{\tau}$=(a,b,c) and a,b, and c are unknown real numbers to be determined.

The essence of the direct variational method for solving the inverse problem is to determine the unknown parameters a,b and c using the discrete least-square approximation:

$$\text{Min}\ \ H(\bar{\tau}) = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[u\left(x_i, y_j, z_k\right) - u_a(x_i, y_j, z_k, \bar{\tau})\right]^2 \tag{20}$$

where $u\left(x_i, y_j, z_k\right)$ are the given values of the solution of problem (1)-(3) in the region D at the point $\left(x_i, y_j, z_k\right)$ in the set Γ, while $u_a(x_i, y_j, z_k, \bar{\tau})$ are the values of the approximate solutions of the same problem, but in the region D ($\bar{\tau}$) evaluated at the same point $\left(x_i, y_j, z_k\right)$ of Γ.

Relation (20) is unconstrained non-linear optimization problem for the unconstrained Hooke and Jeeves method used (3) & (4).

The direct variational method for solving the Inverse Interior Problem of Helmholtz Equation:-

The direct variational method is used for solving the inverse interior Helmholtz equation. In this case to calculate the objective function $H(\bar{\tau})$ one must solve the linear partial differential equation

$$u_{xx} + u_{yy} + u_{zz} + k^2u^2 = f \ \ in\ D(\bar{\tau}), \tag{21}$$

where f= f(x,y,z) is defined in (12),with the boundary condition

$$u=0 \text{ on } D(\bar{\tau}). \tag{22}$$

The variational method as discussed in section (1) for the solution of the direct problem is used for solving equation (21)-(22), at any iteration $\boldsymbol{\ell}$ of the Hooke and Jeeves method given solution $u_a(x_i, y_j, z_k, \bar{\tau}^{\ell})$ in the region D($\bar{\tau}$). This solution together with the given solution u(x,y,z) are evaluated at each point (x_i, y_j, z_k) in the set $\Gamma \subset D$. To calculate the objective equation (20) at $\bar{\tau} = \bar{\tau}^{\ell}$, It is important to mention that the Hooke and Jeeves method is used in this work.

Firstly it is used to minimize the objective equation (20),to find the vector of the unknowns $\bar{\tau} = (a, b, c)$, at any iteration $\boldsymbol{\ell}$ of this method. Secondly the method is also used simultaneously to find the solution $u_a(x_i, y_j, z_k, \bar{\tau}^{\ell})$ of equation (21)-(22) in the region D($\bar{\tau}^{\ell}$).

It is too important to refer here for the Hooke and Jeeves method (to find the unknown coefficients in the solution $u_a$at any initial $\boldsymbol{\ell}$ in the first use of the Hooke and Jeeves method for solving the optimization equation (20).

Numerical Simulation:

To avoid the need for experimental measurements, that is hard to achieve, the following numerical simulation is used to check the usefulness of any new technique. The results of the direct problem which is considered in section (2.4) are used as a given values of the solution (measurements) at each point of the set Γ which is defined in the form:

$$\Gamma = \{(x_i, y_j, z_k) | -1 \le x_i \le 1, h_1(x_i) \le y \le h_2(x_i), -g_1(x_i, y_j) \le z_k \le g_2(x_i, y_j) \text{ for }$$
$$i = 1,2, \dots, I, j = 1,2, \dots, J, k = 1,2, \dots K\}$$

where

$$h_2(x_i) = \sqrt{2.25 - 1.125x_i^2}, \qquad h_1(x_i) = -h_2(x_i)$$
$$g_2(x_i, y_j) = 4(1 - x_i^2 - {y_j}^2/2.25) \text{ and } g_1(x_i, y_j) = -- g_2(x_i, y_j).$$

To solve the inverse the unknown region D is approximated by the unknown region D($\bar{\tau}$) defined in equation (19) where $\bar{\tau} = (a, b, c)$ and a, b unknowns to be determined.

These unknown parameters are determined by using the direct variational method that depends on minimizing (20) with respect to these unknown, and which are solved by using the unconstrained Hooke and Jeeves method. A computer program is used to solve the interior problem. In this program the non-linear optimization equation (20) is solved for different initial values of the unknowns a,b and c with each initial values the initial step lengths h=0.1,h=0.5 and h=1 are used 9in the method of Hooke and Jeeves.

Table (2), The results for a,b and c the approximate values of the coefficients a_n,(n=1,2,…,N) in the approximation solution equation (9) with N=10 are tabulated below: a=1,b=1.5,c=2, $x_n = \sqrt{b^2 - \left(\frac{b_n}{a}\right)^2}$

n	x_n
1	0.998968
2	5.420926×10^{-8}
3	2.9417344×10^{-8}
4	1.853727×10^{-7}
5	8.5010249×10^{-3}
6	-4.1453167×10^{-5}
7	-2.9147264×10^{-5}
8	-4.298106×10^{-4}
9	-17257731×10^{-3}
10	1.2229377×10^{-3}

Remarks and Conclusions

1.The initial values of the unknown parameters can be chosen such that the region which is obtained using these initial values contains the given points that the solution of this problem.

2. The convergence of the numerical method is ensured by the convergence of the direct Ritz method for variational problem, and by the convergence of the algorithm used to solve the non-linear programming problem.

3. The numerical integration method in the problem has been carried at using the Gaussian quadrature integration method of order 7.

4. The inverse problem is solved for different initial values of the unknown a, b &c and for each such initial values, different step lengths in the optimization method of Hooke and Jeeves are used. The approximate values of the above unknown coincide with their corresponding exact values. And for simplicity, we give the results of one execution.

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