On Sm-Modules

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Abstract

Let R be a commutative ring with identity, and let M be a unitary R-module. We introduce a concept of sm-module as follows: M is called sm-module if and only if $\sqrt{ann_R N}$ is a semimaximal ideal of R, for each maximal submodule N of M.

In this paper, some properties and characterizations of sm-modules is given also, various basic results a bout sm-module are considered. Moreover, some relations between sm-modules and other types of modules are considered.

On Sm-Modules

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Introduction

Every ring considered in this paper will be assumed to be commutative with identity and every module is unitary. We introduce the following: An R-module M is called an sm-module if and only if $\sqrt{ann_R N}$ is a semimaximal ideal of R, for each maximal submodule N of M, where annRN={r:r∈R and rN=0}.

Our concern in this paper is to study sm-modules and to look for any relation between sm-modules and certain types of well-known modules. This paper consists of two sections. Our main concern in section one is to define and study sm-modules. We introduce some characterizations for this concept. Also, other basic results about this concept are given. In section two, we study the relation between sm-modules and max-modules, multiplication modules, bounded modules and with the other types of modules.

1. Basic Properties of sm-modules

In this section, we introduce the concept of sm-module and give some characterization and properties of this concept; we end this section by study the relationships between sm-modules and semisimple rings.

We start with the following definition.

1.1 Definition

An R-module M is called sm-module if and only if $\sqrt{ann_R N}$ is a semimaximal ideal of R for each maximal submodule N of M. Specially, a ring R is called sm-ring if and only if R is sm-R-module.

Recall that an ideal I of a ring R is said to be semimaximal ideal if I is an intersection of finitely many maximal ideals of R, [1,Def.(1.2.1),p.16].

1.2 Examples and Remarks

1. Z_6 as a Z-module is sm-module. In general Z_n as a Z-module is sm-module, where n is a positive integer and not prime number.
2. Let p be a prime number. Then the Z-module Z_p is not sm-module.
3. Z as a Z-module is not sm-module. Since pZ is maximal submodule for each p, p is prime number and $\sqrt{\text{ann}_Z(pZ)} = \sqrt{0} = (0)$ for each p. Hence (0) is not semimaximal ideal of Z.
4. Every maximal submodule of an sm-module is an sm-module.

Proof: Let K be a maximal submodule of M. Then $\sqrt{\text{ann}_R K}$ is semimaximal ideal of R (since M is sm-module). To show that K is sm-module. Let N be a maximal submodule of K. Since N⊆K, then $\text{ann}_R K \subseteq \text{ann}_R N$which implies that $\sqrt{\text{ann}_R K} \subseteq \sqrt{\text{ann}_R N}$, but $\sqrt{\text{ann}_R K}$ is semimaximal ideal of R. Thus by [1, Prop.(1.2.11),p.20], $\sqrt{\text{ann}_R N}$ is semimaximal ideal of R and hence K is an sm-module.

5. Let $M=\underset{p}{\oplus} Z_p$ as a Z-module. Then M is not sm-module.
6. Q as a Z-module is not sm-module.
7. The homomorphic image of sm-module is not sm-module.

   For example:Z_6 as a Z-module is an sm-module. Define $f:Z_6 \longrightarrow \frac{Z_6}{(\bar{2})}$, $f(n) = n + (\bar{2})$ fo all $n \in Z_6$. It is easily proved that f is homomorphism, but $\text{ann}_Z \frac{Z_6}{(\bar{2})} \cong Z_2$, Z_2 is not sm-module by (2).
8. Let $M=Z\oplus Z_n$ be a Z-module, n is any positive integer is not sm-module.
9. Recall that an R-module M is said to be max-module if $\sqrt{\text{ann}_R N}$ is a maximal ideal of R, for each non-zero submodule N of M, [2,Def.(2.1),p.4].

The following proposition shows that the class of sm-modules containing in the class of max-modules.

1.3 Proposition

Every max-module is sm-module.

Proof: Let M be a max-module. Then for each non-zero submodule N of M, $\sqrt{ann_R N}$ is maximal ideal of R. Hence by [1, Ex. and Rem.(1.2.2)(2),p.16], $\sqrt{ann_R N}$ is semimaximal ideal of R for each non-zero submodule N of M. Therefore $\sqrt{ann_R N}$ is semimaximal ideal of R for each maximal submodule N of M and hence M is sm-module.

Note that, the converse of proposition (1.3) is not true in general. For example, the Z-module $M=Z_2\oplus Z_{20}$ is sm-module, but is not max-module. Since $N_1=(\bar{0})\oplus(\bar{2})$ and $N_2=(\bar{0})\oplus(\bar{5})$ are maximal submodules of M. Then $\sqrt{ann_Z N_1}=\sqrt{Z\cap 10Z}=\sqrt{10Z}=10Z$ is semimaximal ideal of R and $\sqrt{ann_Z N_2}=\sqrt{Z\cap 4Z}=\sqrt{4Z}=2Z$ is semimaximal ideal of R, which implies M is sm-module. But $\sqrt{ann_R N_1}$ is not maximal ideal of R. Thus M is not max-module.

The class of sm-modules is closed under direct sum as the following result shows.

1.4 Proposition

Let M_1, M_2 be two sm-R-modules. Then $M_1\oplus M_2$ is also sm-R-modules.

Proof: Let $N=N_1\oplus N_2$ be a maximal submodule of M, where N_1, N_2 are maximal submodules of M_1 and M_2 respectively. Then $\sqrt{ann_R N}=\sqrt{ann_R(N_1\oplus N_2)}=\sqrt{ann_R N_1\cap ann_R N_2}=\sqrt{ann_R N_1}\cap\sqrt{ann_R N_2}$, but $\sqrt{ann_R N_1}$ and $\sqrt{ann_R N_2}$ are semimaximal ideals of R (since M_1 and M_2 are two sm-modules). Thus by [,Prop.(1.2.14),p.21], $\sqrt{ann_R N}$ is semimaximal ideal of R and hence $M= M_1\oplus M_2$ is sm-module.

So, we have the following application of (1.4).

1.5 Corollary

Let M_α be an sm-R-modules for all α. Then $\underset{\sim\in\wedge}{\oplus} M_\alpha$ is an sm-module.

The following corollary is a special case of proposition (1.4).

1.6 Proposition

Let M be an R-module. If M is sm-module, then M^2 is also sm-module.

Proof: It is clear that $M^2= M\oplus M$. So according to proposition (1.4), M^2 is an sm-module.

1.7 Remark

It is not necessary that every direct summand of sm-module is sm-module, for example: $M=Z_2\oplus Z_{20}$ as a Z-module is sm-module, but the Z-module Z_2 is not sm-module.

Recall that an R-module M is said to be divisile if and only if rM=M for every non-element r in R, [3].

By using this concept, we have the following.

1.8 Proposition

Let M be an R-module, if M is sm-module and every submodule N of M is divisible, then $\sqrt{ann_R N} = \sqrt{ann_R rN}$ for each maximal submodule N of M such that $rN\neq(0)$, $r\in R$.

Proof: It is abvious.

The following results are another characterizations of sm-module, but first we need to recall some definitions.

An R-module M is called semisimple if every submodule of M is a direct summand of M. And a ring R is said to be semisimple ring if and only if R is a semisimple R-module, [3].

A ring R is a Boolean ring in case each of its elements is an idempotent, [4].

Next, we have the following proposition.

1.9 Proposition

Let M be an R-module. Then M is sm-module if and only if $R/\sqrt{ann_R N}$ is semisimple ring for each maximal submodule N of M.

Proof: If M is sm-module, then $\sqrt{ann_R N}$ is semimaximal ideal of R. Thus by [1,Prop.(1.2.5),p.17], $\frac{R}{\sqrt{ann_R N}}$ is semisimple ring.

Conversely, if $\frac{R}{\sqrt{ann_R N}}$ is semisimple ring, then by [1, Prop.(1.2.5),p.17], $\sqrt{ann_R N}$ is semimaximal ideal of R. Thus M is sm-module by def. (1.1).

Now, we deduce the following corollaries.

1.10 Corollary

Every module M over semisimple ring R is sm-module.

Proof: The result follows directly from [1, Rem.(1.1.34)(3),p.9] and prop.(1.9).

It is known that if R is a Boolean ring, then every proper ideal of R is semimaximal ideal, [1,Cor.(1.2.7),p.18].

1.11 Corollary

Every module M over a Boolean ring is sm-module.

Proof: It is clear that $\sqrt{ann_R N}$ is a proper ideal of R for each maximal submodule N of M. Then the result follows from [1,Cor.(1.2.7),p.18] and prop.(1.9).

2. Modules Related to sm-modules

In this section, we study the relationships between sm-modules and multiplication, bounded, uniform, projective Z-regular and prime modules.

We note that if M is sm-module, then it is not necessary that R is sm-ring, for example: The Z-module Z_4 is sm-module, but Z is not sm-ring. Moreover if R is sm-ring and M is an R-module, then M is not necessarily sm-module, for example: Consider the Z_6-module Z_2, Z_6 is sm-ring, but Z_2 is not sm-module.

Recall that an R-module M is called faithful R-module if $ann_R M=0$, [4].

An R-module M is said to be multiplication module if for every submodule N of M, there exists an ideal I of R such that N=IM, [5].

However, in the class of the faithful multiplication module, they are equivalent as the following result shows.

2.1 Proposition

If M is a faithful multiplication R-module, then M is sm-module if and only if R is sm-ring.

Proof: If M is sm-module. To show that R is sm-module, let I be a maximal ideal of R. Then IM is maximal submodule of M. Hence N=IM is a maximal submodule of M. Thus $\sqrt{ann_R N}$ is a semimaximal ideal of R because M is sm-module. On the other hand, since M is faithful multiplication R-module, then $ann_R N= ann_R I$, so $\sqrt{ann_R N} = \sqrt{ann_R I}$. Thus $\sqrt{ann_R I}$ is a semimaximal ideal and R is a sm-ring.

Conversely, if R is sm-ring. To show that M is sm-module, let N be a maximal submodule of M. Since M is a multiplication R-module, N=IM for some ideal I of R. But M is faithful, $ann_R N= ann_R IM= ann_R I$ and so $\sqrt{ann_R N} = \sqrt{ann_R IM} = \sqrt{ann_R I}$ which is a semimaximal ideal of R. Therefore M is an sm-module.

Recall that an R-submodule N of M is called essential in M if for each non-zero R-submodule L of M, $N \cap L \neq 0$, [5].

Now, we can give the following result:

2.2 Proposition

Let M be an R-module and let $0 \neq x \in M$ such that:

1. Rx is an essential submodule of M.

2. $\sqrt{ann_R(x)}$ is a semimaximal ideal of R, and

3. $\sqrt{ann_R M} = \sqrt{ann_R(x)}$.

Then M is an sm-module.

Proof: Let N be a maximal submodule of M. Since Rx is an essential submodule of M, there exists $0 \neq t \in R$ such that $0 \neq tx \in N$ and hence $(tx) \subseteq N$. This implies that $ann_R N \subseteq ann_R(tx)$ and so $\sqrt{ann_R N} \subseteq \sqrt{ann_R(tx)}$. But $N \subseteq M$, then $\sqrt{ann_R M} \subseteq \sqrt{ann_R N}$ and hence $\sqrt{ann_R(x)} \subseteq \sqrt{ann_R N}$ by (condition 3). Thus,

$\sqrt{ann_R(x)} \subseteq \sqrt{ann_R N} \subseteq \sqrt{ann_R(tx)}$ …(1)

Let $r \in \sqrt{ann_R(tx)}$, then $r^n tx=0$ for some $n \in Z^+$ and $r^n t \in ann_R(x)$. But $tx \neq 0$; that is $t \notin annR(x)$ and by (condition 2) $\sqrt{ann_R(x)}$ is semimaximal ideal of R, so $r \in \sqrt{ann_R(x)}$. Thus

$\sqrt{ann_R(tx)} \subseteq \sqrt{ann_R(x)}$ …(2)

Thus by (1) and (2), $\sqrt{ann_R(tx)} = \sqrt{ann_R(x)}$ and so $\sqrt{ann_R N} = \sqrt{ann_R(x)}$. Therefore (by condition 2) $\sqrt{ann_R N}$ is a semimaximal ideal of R and M is an sm-module by def. (1.1).

An R-module M is called uniform if every non-zero R-submodule of M is essential [5].

So, we have that following application of (2.2).

2.3 Corollary

Let M be a uniform R-module such that $\sqrt{ann_R(x)}$ is semimaximal ideal of R and $\sqrt{ann_R M} = \sqrt{ann_R(x)}$ for some $x \neq 0$. Then M is sm-module.

An R-module M is said to be bounded module if there exists an element $x \in M$ such that $ann_R M = ann_R(x)$ [6].

By using this concept, we have the following.

2.4 Proposition

If M is a bounded R-module such that $\sqrt{ann_R(x)}$ is a semimaximal ideal of R for some $0 \neq x \in M$, then M is an sm-module.

Proof: We have M is bounded, then $ann_R M = ann_R(x)$ for some $0 \neq x \in M$ and so $\sqrt{ann_R M} = \sqrt{ann_R(x)}$ for some $0 \neq x \in M$. Therfore by coro.(2.3), M is sm-module.

The following results are another consequences of proposition (2.4), but first we need to recall the definition of projective module.

An R-module M is called projective if for every R-module epimorphism $h: A \longrightarrow B$ and $f \in Hom_R(M,B)$, there exists $g \in Hom_R(M,A)$ such that $h \circ g = f$ [3,p.217].

2.5 Corollary

Let R be an integral domain. Then every projective R-module M such that $\sqrt{ann_R(x)}$ is semimaximal ideal of R for some $0 \neq x \in M$ is an sm-module.

Proof: According to [6,Coro.(1.1.12),p.10] and proposition (2.4).

2.6 Corollary

Let M be a cyclic R-module such that $\sqrt{ann_R(x)}$ is semimaimal ideal of R for some $0 \neq x \in M$. Then M is sm-module.

Proof: The result is directly by [6, Coro.(1.1.3),p.7] and proposition (2.4).

Recall that an R-module M is called regular if given any element m in M, there exists $f \in M^*$ such that $m = f(m)m$ where $M^* = Hom_R(M,R)$, [3].

The J-radical J(N) of a submodule N of an R-module M is defined as the intersection of all maximal submodules containing N; that is $J(N) = \cap\{P \subseteq M : P \text{ is maximal and } N \subseteq P\}$, [7].

By using these concepts, we have the following.

2.7 Proposition

Let M be a regular multiplication R-module and N be a submodule of M. Then the following statements are equivalent:

1. M is an sm-module.

2. $\sqrt{ann_R(J(N))}$ is semimaximal ideal of R.

3. $R/\sqrt{ann_R(J(N))}$ is semisimple ring.

Proof: (1) $\Rightarrow$ (2): Let M be an sm-module. Then $\sqrt{ann_R N}$ is semimaximal ideal of R for each maximal submodule N of M. Thus by [7,Prop.(2.3),p.4], J(K)=K for every submodule K of M which implies that J(N)=N for every maximal submodule N of M and hence $\sqrt{ann_R(J(N))}$ is semimaximal ideal of R.

(2) $\Rightarrow$ (3) It is abvious according to [,Prop.(1.2.5),p.17].

(3) $\Rightarrow$ (1) It follows directly by proposition (1.).

Next, the following definitions are needed.

An R-module M is said to be a prime module if $ann_R M=ann_R N$ for every non-zero submodule N of M, [8].

An R-module M is called quasi-maximal module if and only if $\sqrt{ann_R M}$ is semimaximal ideal of R, [9].

However, in the class of prime module the two concepts sm-module and quasi-maximal module are equivalent as the following result shows.

2.8 Proposition

Let M be a prime R-module. Then M is sm-module if and only if M is quasi-maximal module.

Proof: If M is sm-module, then $\sqrt{ann_R N}$ is semimaximal ideal of R for each maximal submodule N of M. Hence $ann_R M = ann_R N$ (since M is prime module). Therefore $\sqrt{ann_R M} = \sqrt{ann_R N}$, which implies that $\sqrt{ann_R M}$ is semimaximal ideal of R and hence M is quasi-maximal module.

Conversely, let M be a quasi-maximal module. Then $\sqrt{ann_R M}$ is semimaximal ideal of R. Now, for every non-zero submodule N of M, $ann_R M = ann_R N$ because M is prime module. Then $\sqrt{ann_R M} = \sqrt{ann_R N}$ for every non-zero submodule N of M, but $\sqrt{ann_R M}$ is semimaximal ideal of R, which implies that $\sqrt{ann_R N}$ is semimaximal ideal of R for every non-zero submodule N of M and hence $\sqrt{ann_R N}$ is semimaximal ideal of R for each maximal submodule N of M, which completes the proof.

The condition M is prime can not be dropped from proposition (2.8) as the following examples shows.

2.9 Example

Consider $M=Z_2 \oplus Z_3$ as a Z-module. M is not prime Z-module.

M is quasi-maximal module. Since $\sqrt{ann_Z M} = \sqrt{ann_Z (Z_2 \oplus Z_3)} = \sqrt{6Z} = 6Z$ is semimaximal ideal of Z. But M is not sm-module. Since $(\bar{0})$ is the only maximal submodule of M and $\sqrt{ann_Z (\bar{0})} = \sqrt{Z} = Z$ is not semimaximal ideal.

The following results are another consequences of proposition (2.8), but first we need to recall some definitions

An R-module M is said to be searial R-module if the R-submodules of M are linearly with respect to inclusion, [8].

An R-module M is called Z-regular if for all $m \in M$, there exists $f \in Hom_R(M,R)=M^*$ such that f(m)m=m, [6].

An R-module M is called semiprime if and only if $ann_R N$ is a semiprime ideal of R for each non-zero R-submodule N of M, [10].

Hence, we have the following consequences of (2.8).

2.10 Corollary

Let M be a Z-regular serial R-module. Then M is sm-module if and only if M is quasi-maximal module.

Proof: From [10,Prop.(4.2.2),p.71], [10,Prop.(4.2.1),p.70], we get M is prime module and hence by proposition (2.8) we get the result.

2.11 Corollary

Let M be a uniform semiprime R-module. Then M is quasi-maximal module if and only if $\frac{R}{\sqrt{\operatorname{ann}_R N}}$ is semisimple ring for each maximal submodule N of M.

Proof: If M is quasi-maximal module. Then the result follows from [10,Prop.(4.2.3),p.72], proposition (2.8) and proposition (1.9).

Conversely, If $\frac{R}{\sqrt{\operatorname{ann}_R N}}$ is semimaximal ring for each maximal submodule N of M. Thus by proposition (1.9), we get M is sm-module and hence the result follows according to [10,Prop.(4.2.3),p.72] and proposition (2.8).

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