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Some Results of Cubic Family of Pseudo Differential Operators

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Abstract

In this paper, we recall the results related to the quadratic family of pseudo-differential operators, and we gave similar results for to the cubic family of pseudo-differential operators, also we benefited from the facilities obtained by using the pseudo-differential operators. We define the pseudo-differential operator on R^n by using some particular kind of functions known as symbol with some examples. At the end, we explain the hypotheses on the class of symbols and minimal decreasing ray to prove the existence of a total system of eigenvectors.

Keywords: Pseudo differential operator, nonlinear eigenvalue problem, class of Schatten, Minimal decreasing ray, total system of eigenvectors.

بعض النتائج للعائلة الثلاثية للمؤثرات شبه التفاضلية

فاطمة ياسين طه و فاطمة محمد عبود

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الخلاصة

في بحثنا هذا نعطي تذكير للنتائج المتعلقة بالعائلة التربيعية للمؤثرات التفاضلية الزائف، كما نقدم نتائج مماثلة للعائلة المكعبة للمؤثرات التفاضلية الزائفة وبرهنا نتائجنا بالاستفادة من التسهيلات الناتجة من استخدام المؤثرات التفاضلية الزائفة. كما

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قمنا بتعريف مؤثرات التفاضل الزائفة على R^n باستخدام نوع معين من الدوال يعرف بالرمز مع إعطاء بعض الأمثلة التوضيحية. في النهاية نقدم شرحا للفرضيات المتعلقة برموز المؤثرات التفاضلية الزائفة ونقوم بتعريف الاشعة ذات التناقص الادنى لأجل التمكن من برهنة وجود نظام تام من المتجهات الذاتية.

الكلمات المفتاحية: المؤثر التفاضلي الزائف، مشكلة القيمة الذاتية غير الخطية، الشعاع المتناقص الأدنى، النظام الكلي للمتجهات الذاتية.

Introduction

Pseudo-differential operators developed from the theory of singular integral operators, and it provides a unified treatment of differential and integral operators. They are based on the intensive use of the Fourier transformation and its inverse.

The linear pseudo differential operators can be characterized by generalized Fourier multipliers, called symbols. The class of pseudo differential operators form an algebra, and the operations of composition, transposition and adjoining of operators can be analyzed by algebraic calculations of the corresponding symbols, which permit us to have a facilities in the treatment of this kind of operators.

Moreover, this class of pseudo differential operators is invariant under diffeomorphic coordinate transformations. As linear mappings between distributions. For elliptic pseudo differential operators, we construct parametrices. For elliptic differential operators, in addition, we construct a fundamental solution if they exist.

1. Basic concepts of pseudo-differential Operators

In this section we recall some well-known concepts and results.

Definition 1.1.[1] : The scalar λ is called an eigenvalue of A, and non zero vector x is an eigenvector of A corresponding to λ, if

$$Ax = \lambda x$$

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has nontrivial solution x, where A is $n \times n$ matrix. The set of all eigenvalues of matrix A is called the spectrum of A, and is denoted by $\sigma\ (\boldsymbol{A})$.

Definition 1.2.[2] : let p,ς and η be real numbers with $0 \leq \eta \leq 1$, $0 \leq \varsigma \leq 1$. As example the class $S^{p}_{\varsigma,\eta}(X \times R^n)$ consists functions $a(x,\theta)\ \in C^{\infty}(x \times R^n)$ such that for any multi-indices α, β and any compact set K⊂ X a constant $C_{\alpha,\beta,k}$ exists for which

$$|\partial_\theta^\alpha \partial_x^\beta a(x,\theta)| \leq C_{\alpha,\beta,k} \langle\theta\rangle^{m-\varsigma|\alpha|+\eta|\beta|}$$

Where x ∈ K and $\theta \in R^n$ are constants.

Instead of $S^{p}_{1,0}(X \times R^n)$ we just write $S^p(X \times R^n)$. Furthermore, we will occasionally simply write $S^{p}_{\varsigma,\eta}(X \times R^n)$ instead of $S^{p}_{\varsigma,\eta}$. We also used $S^{-\infty}=\cap_p S^p$.

Definition 1.3. [3]:

First we explain that

$$Pu(x) = (2\pi)^{-n} \int\ e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi\ . \qquad (1.1)$$

The function p is called the symbol of the operator P. Pseudo-differential operators are a generalization of differential operators in that they are defined by symbols which are non-necessarily polynomials with respect to ξ. Let us introduce the symbols that we shall consider.

Definition 1.4 [3]: Given $u \in \mathcal{S}$ and $a \in S^{\mu,m}$, we set

$$a(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi\ . \qquad (1.2)$$

The operator $a(x,D)$ is called the pseudo-differential operator of symbol a and will also be denoted by

$$Op(a) := a(x,D)$$

Definition 1.5 (Non-linear eigenvalue)[4]: If $T\mathrm{x} = \lambda \mathrm{x}$ for a non-nul vector x , a complex number λ is referred to be an eigenvalue of T for a linear eigenvalue problem. In the case of a non-linear eigenvalue problem, if the problem $T(\lambda)x = 0$ has a non-trivial solution x, λ is said to be a non-linear eigenvalue of T. When $T(\lambda) = \lambda^m 1 + \lambda^{m-1} H_{m-1} + \cdots + \lambda H_1 + H_0$ where $H_0, \dots, H_{m-1}$ are some operators .

Definition 1.6.[5]: We say that the symbol a (x, ξ) is poly-homogeneous, if a (x, ξ) is of the following form:

$$a(x,\xi) = \sum_{\ell \in \mathbb{N}} a_{M-\ell}(x,\xi)$$

with $a_{(M-l)\,(x,\,\xi)}$ is homogeneous of degree M- ℓ.

Definition 1.7.[5]: We say that the symbol a (x, ξ) is poly-quasi-homogeneous, if a (x, ξ) is of the following form:

$$a(x,\xi) = \sum_{\ell \in \mathbb{N}} a_{M-\ell}(x,\xi)$$

with $a_{(M-l)\,(x,\,\xi)}$ is quasi-homogeneous of degree M- ℓ .

Definition 1.8. [5] : We say that the symbol a (x, ξ) is quasi-elliptical if it is poly-quasi-homogeneous whose main symbol does not vanish outside zero, i.e.:

$$a_M(x,\xi) \neq 0, \ \text{ for } (x,\xi) \in \mathbb{R}^{2n} \setminus \{0\}.$$

Definition 1.9(Schatten classes) [6]: The compact operator T, on a Hilbert space $\boldsymbol{H}$ is in the Schatten class C_p for some p in $[1, +\infty]$, if the sequence μ_j of the eigenvalues of $|T| = \sqrt{T^*T}$

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satisfy $\sum_j \mu_j^p < +\infty$,. When $\mathrm{p} = 1, \mathrm{C}_1$ is referred to as Trace class operators, and when $\mathrm{p} = 2, \mathrm{C}_2$ is referred to as Hilbert -Schmidt class of operators.

The trace map is defined as follows when $\mathrm{p} = 1$:

Operators, and when $\mathrm{p} = 2, \mathrm{C}_2$ referred to as Hilbert -Schmidt class of operators. The trace map is defined as follows when $\mathrm{p} = 1$:

$$C_1 \ni T \leftrightarrow \mathrm{Tr}(T) = \sum_j \langle T\varphi_j \mid \varphi_j \rangle$$

Where (φ_j) is an orthonormal basis. It can be shown that the Trace map is continuous and this is unaffected by the used basis.

$$|\mathrm{Tr(T)}| \leq \| \, |T| \, \|_{c_1}$$

2. Cubic family of Pseudo-differential operators

2.1.[5]: Assumptions on classes of symbols

We consider the quadratic and cubic family of operators, denoted by $\hat{L}_q$, $\hat{L}_c$ respectively:

$$\hat{L}_q(\lambda) = \hat{H}_0 + \lambda \hat{H}_1 + \lambda^2$$

$$\hat{L}_c(\lambda) = \hat{H}_0 + \lambda \hat{H}_1 + \lambda^2 \hat{H}_2 + \lambda^3$$

Where $\widehat{H}_0$,$\hat{H}_1$ and $\hat{H}_2$are pseudo differential operators with symbols $H_0(x,\xi)$, $H_1(x,\xi)$ and $H_2(x,\xi)$, respectively, verify the following hypotheses:

(HC) $H_0(x,\xi), H_1(x,\xi)$ Belong to $\mathcal{C}^{\infty}(\mathbb{R}^{2n})$, and $H_1(x,\xi) \geq 0$.

There are weight functions ϕ, φ, m as:

$(HP-1)$ There are constants C, C' as:

$$C'm(x,\xi) \leq H_0(x,\xi) \leq Cm(x,\xi).$$

$(HP-2)$ For everything $(x,\xi) \in \mathbb{R}^{2n}$, there is a constant $C_{\alpha\beta}$ as

$$\left|\partial_x^\alpha \partial_\xi^\beta H_0(x,\xi)\right| \leq C_{\alpha\beta} m \varphi^{-|\alpha|} \phi^{-|\beta|}$$

$(HP-3)$ For everything $(x,\xi) \in \mathbb{R}^{2n}$, there is a constant $C'_{\alpha\beta}$ as:

$$\left|\partial_x^\alpha \partial_\xi^\beta H_1(x,\xi)\right| \leq C'_{\alpha\beta} m^{\frac{1}{2}} \varphi^{-|\alpha|} \phi^{-|\beta|}.$$

$(HP-4)$ For everything $(x,\xi) \in \mathbb{R}^{2n}$, there is a constant $C''_{\alpha\beta}$ as:

$$\left|\partial_x^\alpha \partial_\xi^\beta H_2(x,\xi)\right| \leq C''_{\alpha\beta} (m(x,\xi))^{\frac{1}{3}} \varphi^{-|\alpha|}(x,\xi) \phi^{-|\beta|}(x,\xi)$$

We note that the hypothesis $(HP-1)$, $(HP-2)$, $(HP-3)$ are satisfied for the symbols $H_0(x,\xi)$, $H_1(x,\xi)$ for the quadratic and cubic families $\hat{L}_q(\lambda)$, $\hat{L}_c(\lambda)$ respectively and the hypothesis $(HP-4)$ is used for the cubic family. The quadratic family was studied by many authors, we pay here more attention for the cubic family.

In the following we give Pham-Robert theorem, which is satisfied for the following polynomial family of operators:

$$\hat{L}(\lambda) = \hat{H}_0 + \lambda \hat{H}_1 + \lambda^2 \hat{H}_2 + \cdots + \lambda^{\mathrm{m}} \hat{H}_m$$

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Which is a non-bounded family defined on a Hilbert space $\mathcal{H}$, λ is a complex number and $\widehat{H}_0, \widehat{H}_1, \ldots, \widehat{H}_m$ are pseudo differential operators with symbols $H_0(x,\xi)$, $H_1(x,\xi), \ldots, H_m(x,\xi)$, respectively. In addition, we have:

$\widehat{H}_0$ Is a self-adjoint positive operator with a domain D $(\widehat{H}_0)$ dense in the Hilbert space $\mathcal{H}$.

There exists a real number p>0 such that $\widehat{H}_0$ belong to the class of Schatten $C^p(\mathcal{H})$.

For every j,$0 \leq j \leq k-1$, the operators $\widehat{H}_0^{-1}\widehat{H}_j$ and $\widehat{H}_j\widehat{H}_0^{-1}$ are bounded in $\mathcal{H}$.

Using these hypothesis on $\hat{L}(\lambda)$ we can give Pham-Robert theorem: Theoream 2.1.1.[5]: (Pham-Robert) We suppose that there exist s half-lines

$\Delta_1, \ldots, \Delta_s$ starting from 0 and dividing the complex plane into s open sectors $\alpha_j < \frac{\pi}{p}$ for j = 1,…,s. We further assume that there exists an integer $N \geq 0$ such that:

$$\|L(\lambda)^{-1}\|_{L(\mathcal{H}, D(H_0))} = \mathcal{O}(|\lambda|^N) \text{ when } |\lambda| \to +\infty, \lambda \in \Delta_1 \cup \ldots \cup \Delta_s$$

Then the vector space generated by the generalized eigenvectors of L is dense in $\mathcal{H}$.

We give the following results, which are a generalization for the results of [5] for the cubic family with the quadratic family.

Proposition 2.1.2.If

$$L_P(\lambda) = -\Delta + (P(x) - \lambda)^{\mathrm{k}},\ x \in \mathbb{R}^n$$

Where P is a homogeneous polynomial of degree $M \geq 2$, k=2,3.

Then the real positive direction is a polynomial decreasing ray for the family $L_P(\lambda)$ for k=2,3.

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Proof: The proof for k=2, is given in [Thesis Fatima ABOUD], and the proof of k=3 is similar to the case k=2.

Proposition 2.1.3. For the family of operators

$L_P(\lambda) = -\Delta + (P(x) - \lambda)^k,\ x \in \mathbb{R}$, k=2,3

Where P is a homogeneous polynomial of order $M \geq 2$, the eigenvector system generalized $L_P(\lambda)$ is total in $L^2(\mathbb{R})$ for k=2,3.

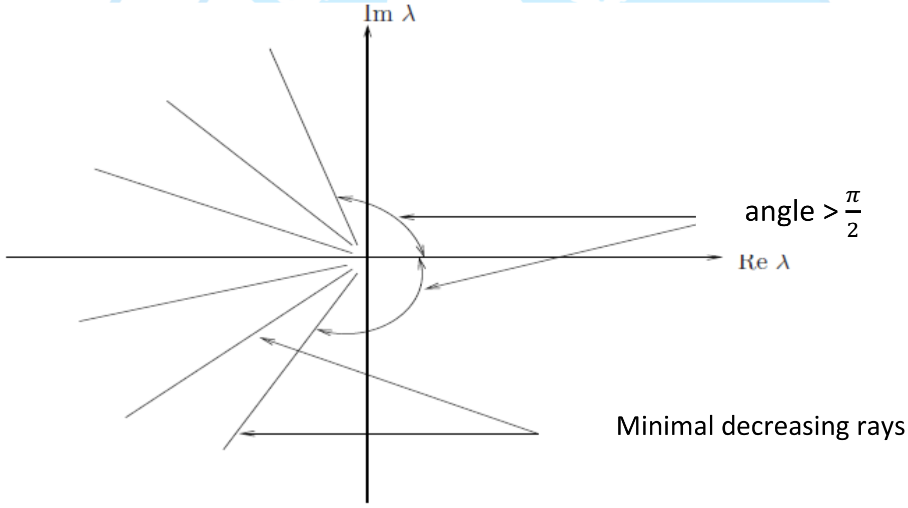

Figure 2.1: Minimal decreasing ray area.

Here we have $1 < p < 2$ for $M \geq 2$ where $p = \frac{M+1}{M} + \epsilon$.

Proof: Since $M \geq 2$, so by using proposition 3.1.1, we obtain that the positive real direction is a decreasing polynomial ray for $L_P(\lambda)$. Hence by Pham-Robert theorem the proof is achieved.

Example 2.1.4.: We consider the cubic family of operators $\widehat{L}(\lambda)$,

$$L_P(\lambda) = -\Delta + (P(x) - \lambda)^3,\ x \in \mathbb{R}^n$$

Which has the following symbol:

$$L(\lambda, x, \xi) = \xi^2 + (P(x) - \lambda)^3$$

Where P is a positive elliptical polynomial. This family is a special case with

$$H_0(x,\xi) = \xi^2 + P^3(x)$$
$$H_1(x,\xi) = -3P^2(x)$$

$$H_2(x,\xi) = 3P(x)$$

Remark 2.1.5.: If $H_0(x,\xi)$ is a quasi-elliptical symbol

Positive and if$H_1(x,\xi)$, $H_2(x,\xi)$ are a quasi-homogeneous symbols, we come back to the previous by a suitable choice of functions φ, ϕ, ϕ'.

2.2 Rays of minimal decreasing

Definition 2.2.1. [5]: For $\theta \in [0,2\pi]$, we say that $\Delta(\theta, \rho_0)$ is rays of decreasing polynomial for the family of operators L (λ) if there is a constant $C > 0$ and an integer $N > 0$ such as:

$$\|L^{-1}(\rho e^{i\theta})\| \le C\rho^N$$

For all $\rho \ge \rho_0$.

Now we study rays of decreasing polynomial for $\hat{L}^{-1}(\lambda)$ for the quadratic and cubic families, we note that the results for quadratic family was already done in [5], we give here the same result for the cubic family.

We set $\lambda = re^{i\theta}$. Then we have

$$L_q(re^{i\theta}, x, \xi) = H_0(x,\xi) + re^{i\theta}H_1(x,\xi) + r^2e^{2i\theta}$$
$$= H_0(x,\xi) + r\cos(\theta)H_1(x,\xi) + r^2\cos(2\theta) + i(r\sin(\theta)H_1(x,\xi) + r^2\sin(2\theta)$$

$$L_c(re^{i\theta}, x, \xi) = H_0(x,\xi) + re^{i\theta}H_1(x,\xi) + r^2e^{2i\theta}H_2(x,\xi) + r^3e^{3i\theta}$$
$$= H_0(x,\xi) + r\cos(\theta)\,H_1(x,\xi) + r^2\cos(2\theta)\,H_2(x,\xi) + r^3\cos(3\theta) + i(r\sin(\theta)H_1(x,\xi) + r^2\sin(2\theta)H_2(x,\xi) + r^3\sin(3\theta))$$

we take note that $rH_1 \geq 0, r^2H_1 \geq 0$, we thus obtain for $|\theta| \leq \frac{\pi}{4}$

$$\left|L_q(re^{i\theta}, x, \xi)\right|^2 \geq H_0^2 + r^4,$$

$$\left|L_{c.}(re^{i\theta}, x, \xi)\right|^2 \geq H_0^2 + r^6,$$

We then deduce that any direction θ such that

$$|\theta - \pi| \leq \frac{\pi}{4} \qquad (2.1)$$

Is a direction of minimal decreasing (cf. fig. 2.1) in the sense that

$$\| \widehat{L_q}(\lambda) \|_{L^2 \to L^2} \leq \frac{C}{1 + |\lambda|^2}$$

$$\| \widehat{L_c}(\lambda) \|_{L^2 \to L^2} \leq \frac{C}{1 + |\lambda|^3}$$

For $\lambda = re^{i\theta}$ and θ verifying (2.1).

We give the following result which is a generalization of Proposition 2.1.2 for the following family of operators:

$$L_p(\lambda) = (-\Delta)^{\ell} + (P(x) - \lambda)^{\mathrm{k}},\ x \in \mathbb{R}^n \qquad (2.2)$$

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Where P is a quasi-homogeneous polynomial of order M and of type$(k_1,\cdots,k_n), k=2,3$.

Proposion 2.2.2.: If $L_P(\lambda)$ is the family in (2.2). So the real direction

Positive is a polynomial decreasing ray for this family.

Now, we give the following examples and we apply our results for them.

Proof: the proof for k=2, is given in [Thesis Fatima ABOUD], and the proof of k=3 is similar to the case k=2.

Example 2.2.3.:

1- If $L_p(\lambda)$ the following family of operators:

$$L_P(\lambda)=\left(-\Delta_{x,y}\right)^4+(|x|^8+|y|^2-\lambda)^2, x\in\mathbb{R}^2, y\in\mathbb{R}$$

We have

$$H_0(x,y,\xi,\eta)=|\xi|^8+|\eta|^8+(|x|^8+|y|^2)^2,$$
$$H_1(x,y,\xi,\eta)=-2(|x|^8+|y|^2)^2,$$

Where $(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}, H_0, H_1$ and H_2 are of order 2 and 1 respectively and of type $\left(\frac{1}{8},\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)$.

For the rays of minimal decreasing of this family we have

$$S_L=\{\lambda\in\mathbb{C}: \Re(\lambda)\geq 0\}$$

Because

$$|\xi|^8+|\eta|^8+(|x|^8+|y|^2-\lambda)^2=0,$$

From where

$$\lambda=|x|^8+|y|^2-\mathrm{i}\sqrt{|\xi|^8+|\eta|^8}\,.$$

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So the minimum decreasing rays of L_λ are in the half-plane:

$$\{\lambda \in \mathbb{C}: \Re(\lambda) < 0\}$$

We have $H_0^{-\frac{1}{2}} \in \mathcal{C}_p$ with p $> \frac{7}{4}$, we then have the following angle condition:

$$\theta < \frac{4\pi}{7}$$

According to Proposition 2.2.2, the positive real axis is a polynomial, hence by using the theorem of Pham-Robert, the system of vectors generalized own of $L_P(\lambda)$ is total in $L^2(\mathbb{R}^n)$.

2- If $L_p(\lambda)$ the following family of operators:

$$L_P(\lambda) = \left(-\Delta_{x,y}\right)^3 + (|x|^8 + |y|^2 - \lambda)^3, x \in \mathbb{R}^2, y \in \mathbb{R}$$

We have

$$H_0(x, y, \xi, \eta) = |\xi|^6 + |\eta|^6 + (|x|^8 + |y|^2)^3,$$
$$H_1(x, y, \xi, \eta) = -3(|x|^8 + |y|^2)^2,$$

$$H_2(x, y, \xi, \eta) = 3(|x|^8 + |y|^2)$$

Where $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}, H_0, H_1$ and H_2 are of order 3, 2 and 1 respectively and of type $\left(\frac{1}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. For Rays of minimal decreasing of this family we have, the whole S_L is

$$S_L = \{\lambda \in \mathbb{C}: \mathrm{Im}(\lambda) = 0\}$$

Because

$$|\xi|^6 + |\eta|^6 + (|x|^8 + |y|^2 - \lambda)^3 = 0,$$

From where

$$\lambda = |x|^8 + |y|^2 - \sqrt[3]{|\xi|^6 + |\eta|^6}\ .$$

So the minimum decreasing rays of L_λ are in the half-plane:

$$\{\lambda \in \mathbb{C}: \mathrm{Im}(\lambda) = 0\}$$

We have $H_0^{-\frac{1}{2}} \in \mathcal{C}_p$ with $p > \frac{7}{4}$, we then have the following opening condition:

$$\theta < \frac{4\pi}{7}$$

According to Proposition 2.2.2, the positive real axis is a polynomial ray for this cubic family, hence by using the theorem of Pham-Robert, the system of vectors generalized of $L_P(\lambda)$ is total in $L^2(\mathbb{R}^n)$, for k=2,3, i.e. for the quadratic and cubic families.

Conclusion

We conclude that for the cubic family of operators we can obtain a similar results that already obtained for the quadratic family. So that we obtain a total system of generalized eigenvectors in the domain of our family of operators.

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