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On Some Properties of Hollow-Lifting-Quasi-Discrete Modules

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Abstract

A notion of the Hollow-Lifting-Quasi-Discrete modules can constitute avery important situation in the module theory. In this paper we will present a key role in some properties and characterizations of Hollow-Lifting-Quasi-Discrete Modules, also we study characterization the relation between lifting property and Hollow-Lifting-Quasi-Discrete Modules. We introduced and study the concept of Hollow-Lifting-Quasi-Discrete Modules a generalization of Hollow-Quasi-Discrete Modules. In this work, we introduce and study classes of concepts which are extremities of Hollow-lifting-Quasi-Discrete modules. We call an R-module M Hollow-Lifting-Quasi-Discrete, if every submodule N of M with $\frac{M}{N}$ hollow-lifting module, there exists a stable direct summand K of M such that $K \leq ce^{N}$ in M. we have the following proper implication:

Hollow-Lifting → Hollow-Lifting-Discrete → Hollow-Lifting-Quasi-Discrete Module.

Keywords: Hollow-Quasi Module, Hollow Lifting-Discrete Module, Hollow-Lifting Module and Hollow-Lifting Quasi.

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بعض خواص مقاسات الرفع المجوفة شبه المتقطعة

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اعدادية الطيبات للبنات – مدينة جلولاء – محافظة ديالى – العراق

الخلاصة

مفهوم مقاسات الرفع المجوفة يمكن ان تشكل حالة مهمة جدا في نظرية الموديل، في هذا البحث سوف نقدم الدور الرئيسي في بعض خواص ومميزات مقاسات الرفع المجوفة وكذلك سوف ندرس مميزات العلاقة بين خاصية الرفع ومقاسات الرفع المجوفة. ونقدم وندرس مفهوم مقاسات الرفع المجوفة شبه المتقطعة كتعميم للمقاسات المجوفة شبه المتقطعة. اي تم عرض ودراسة اصناف من المقاسات التي تكون على طرفي مقاسات الرفع المجوفة، يقال عن المقاس M انه مقاس رفع مجوفة شبه متقطعة اذا كان لكل مقاس مجوف جزئي N في M توجد مركبة جمع مباشر K في N بحيث ان $\frac{M}{N}$ مقاس اجوف شبه متقطع وان $K \leq ce^N$ في M وتعني الحصول على المؤديات التالية:

مقاس رفع اجوف ← مقاس رفع شبه متقطع ← مقاس رفع اجوف شبه متقطع.

الكلمات المفتاحية: مقاس رفع اجوف، مقاس شبه اجوف، مقاس متقطع اجوف و اجوف-متقطع.

Introduction

In this paper all rings R are associative with identity element and the modules are unitary left modulus.

Since, forty years ago the developments of modules with lifting property have been a major area in ring and module theory. Following Oshiro [1], an R-module M is Lifting Module if for each submodule N of M there exists a direct summand M1 such that $M_1 \leq N$ and $\frac{N}{M_1} \ll \frac{M}{M_1}$.

Recently, extremities concepts of extending modules introduced and studies in [2]. The others in [3 and 4] study another generalization of lifting modules, an R-module M is named Hollow-Lifting for each submodule N impels $\frac{\mathrm{M}}{\mathrm{N}}$ is hollow contain coessential submodule which is named a direct summand of M. On other direction, Mehdi S. Abbas and Saad A. Al saadi in [5], introduced the concept strongly lifting modules as a proper stronger of lifting modules. An R-module is called strongly lifting module if every submodule N of M has coessential submodule that is a stable direct summand of M.

Hollow and Hollow Lifting Module

- Hollow, [6 and 7] Anon-zero R-module is named a Hollow if every proper submodule of M is a small.
- Hollow-Lifting module, [8] An R-module M is named Hollow-Lifting module if for each submodule N of M with $\frac{M}{N}$ Hollow, implies a submodule, K′ of M and

$$\mathbf{M = K + K' \text{ such that } K \leq ce\ N}$$

in M with k is a coessential submodule of N in M.
We give many properties related with the term of Hollow-Lifting-Quasi-Discrete Modules as generalization of Hollow-Quasi-Discrete Modules. Firstly, we give the following conditions are for an R-module M.

H_1: for each submodule N of M with $\frac{M}{N}$ is Hollow-Lifting-Quasi-Discrete Module if for each a direct decomposition

$$\mathbf{M = M1 + M2 \text{ such that } M1 \leq N \text{ with } N \cap M2 << M}$$

H_2: when A is a submodule of M that is M / N is a Hollow-Lifting-Quasi-Discrete Modules and isomorphic to a direct summand of M, then A is a direct summand of M.

H_3: The intersection between two direct summand w and u of M with $\frac{M}{w}$, $\frac{M}{u}$ are Hollow-Lifting-Quasi-Discrete Modules such that M = w + u, is again called direct summand of M.

Notes that on R-module M is named Hollow-Lifting- Discrete Module if M satisfiy (H_1) and (H_2) conditions and an R-module M is named Hollow-Lifting-Quasi-Discrete Module if M has (H_1) and (H_3).

- Examples and Remarks

1. Every semi-simple Hollow-Lifting module is called Hollow-Lifting-Quasi-Discrete.

2. Every Hollow-Lifting module is called Hollow-Lifting-Quasi-Discrete Module.

That is for any two direct summands M1 and M2 of M with $\frac{M}{M_1}$ and $\frac{M}{M_2}$ are Hollow - Lifting with

$$\mathbf{M = M_1 + M_2}$$

Proof

suppose that M1 (resp. M2) is either [M resp.(0) or (0) resp. M] with (0) is a direct summand of M, if $M_1 \neq M_2$ then M1 is small in M thus proposition in [2], implies that M1=(0), similary for both cases implies that $M_1 \cap M_2 = 0$ finally Mi = M (i = 1,2) which clear that a direct summand of M.

1. Z-module is not Hollow-Lifting-Quasi-Discrete Module, we can show that Z has no (H1) condition assume that Z has (H1) condition. Assum that $B = 2Z$ a submodule of M and $\frac{Z}{2Z}$ Hollow-Lifting then for each a direct decomposition $Z = M1 + M2$ then $M_1 \leq B$ and $M_2 \cap B << Z$, but Z is indecomposable Z-module then $M = (0)$ and $M_2 = Z$ then $B = M_2 \cap B <<$ Z which is a contradiction since $B + 3Z = Z$ and $3Z \neq Z$.

2. Every Hollow-Lifting discrete module named Hollow-Lifting-Quasi-Discrete Module.

Proof

To see that, let w and u be two direct summand of M with $\frac{M}{w}$ and $\frac{M}{u}$ are Hollow-Lifting and $M = w + u$.

suppose $M = w + A = u + B$ where A,B are submodules of M therefore let $A \cong \frac{M}{w} \cong \frac{w+u}{w} \cong \frac{u}{w \cap u} \cong \frac{w+B}{w \cap u+B}$, and $\frac{w+B}{w \cap u+B}$ is Hollow-Lifting since $\frac{M}{w}$ is Hollow-Lifting, but M has (H2) then

$w \cap u + B$ is named a direct summand. Let $M = (w \cap u + B) + C = (w \cap u) + (B + C)$ where C is asubmodule of M, thus $w \cap u$ is direct summand of M.

3. The converse of (4) in general is not true.

As an example as in (4) a Z-module called Hollow-Lifting-Quasi-Discrete Module since which is Hollow-Lifting, when Z-module does not satisfy Hollow-Lifting -Discrete Module, that is for any non-zero submodule A of Z-module with Z-module / A is a Hollow-Lifting and $\frac{Z}{A} = Z$, but is a direct summands of Z.

4. Every Lifting-Quasi-Discrete Modules is Hollow-Lifting-Quasi-Discrete Modules,

clearly the converse in general is not correct. For example, as in [5] assume that M be an indecomposable R-module which is not implies hollow factor module, it is clear that M is Hollow-Lifting-Quasi-Discrete R-Module, then M is not lifting.

we suppose that M is (H1) and assume that N is a submodule of M that is for each direct decomposition $M = M_1 + M_2$ such that $M_1 \leq N$ with $M_2 \cap N << M$ but M is ndecomposable then $M_1 = (0)$ and $M_2 = M$ Hence $N = M_2 \cap N << M$,thus M is called Hollow-Lifting which a contradiction, therefore, M is not Hollow-Lifting-Quasi-Discrete R-Module.

- Proposition

Let M be satisfy (H_2) condition, if w and u are two Hollow-Lifting-Quasi-Discrete direct summand in M, with $\frac{M}{w}$ and $\frac{M}{u}$ are Hollow-Lifting-Quasi then any epimorphism F: w→u is splits.

Proof:

Assume that $M = w + K$ for a submodule K of M such that $u \cong \frac{w}{\text{Ker F}} \cong \frac{w+K}{\text{Ker f+K}} \cong \frac{M}{\text{Kerf+K}}$ but u is Hollow-Lifting-Quasi-Discrete summand of M , $\frac{M}{\text{Kevf+K}}$ is Hollow-Lifting-Quasi-Discrete Module, hence by (H_2) property of M.

Assume that $K_{erf} + K$ is a direct summand since $K_{erf} \subset W \subset M$ and K_{erf} is a direct summand of M, therefore, in (proposition 3-3), [9] K_{erf} is named a direct summand of w.

- Proposition

Assume that M is an R-Module beside to (H_1) such that the intersection of A and B is coessential in A, then the condition are equivalent when A and B are two submodules of M.

1. M has (H_3).

2. For and two direct summand w, u of M with $\frac{M}{w \cap u}$ is Hollow-Lifting-Quasi-Discrete, $M = w + u$ and $w \cap u << M$ then $M = w + u$.

Proof: 1→2

For each two direct summand w and u in M with $M = w + u$, $\frac{M}{w \cap u}$ is Hollow-Lifting-Quasi-Discrete Modules since $\frac{M}{w \cap u}$ is Hollow-Lifting-Quasi-Discrete Modules then $\frac{M}{w}$ and $\frac{M}{u}$ are Hollow-Lifting-Quasi-Discrete Modules by (proposition 4-3), [10].

By (H_3) property of M such that $w \cap u$ is called a direct summand of M when $w \cap u \leq Ce\ w$, but $w \cap u << M$, we get $w \cap u = 0$, thus $M = w + u$.

Proof: 2→1

Assume that w and u be two direct sum and with $\frac{M}{w}$, $\frac{M}{u}$ are Hollow-Lifting-Quasi-Discrete Modules, then we have $\frac{M}{w \cap u}$ is Hollow-Lifting-Quasi-Discrete Modules by (proposition), [11]. Now, since $M = w + u$, then by (H_1) for each decomposition $M = M_1 + M_2$ then $M_1 \leq w \cap u$ and $w \cap u \cap M_2 << M$. Now, by modular law have $u \cap M = u \cap (M1 + M2) = M1 + (u \cap M2)$ and $M_1 \cap (u \cap M_2) = 0$, hence $u = M_1 + (u \cap M_2)$. Also, $M = w + u = (w + (M_1 + (u \cap M_2)) = w + (u \cap M_2)$ by modular law implies that

$w \cap u \cap M = w \cap u \cap (M_1 + M_2) = M_1 + (w \cap (u \cap M_2))$ and $M_1 \cap (w \cap (u \cap M_2)) = 0$ hence, $M_1 + (w \cap (u \cap M)) = w \cap u$ and, we get w with $u \cap M_2$ are direct summand of M. $M = w + (u \cap M_2)$ for each $m \epsilon M$ we get $m = m_1 + m_2$ for some $m_1 \epsilon M_1$ and $m_2 \epsilon M_2$, therefore, $m \epsilon (w \cap u) + M_2$, thus $M = (w \cap u) + M_2$

- Proposition

Let $M = A + B$ be a Hollow-Lifting-Quasi-Discrete Modules where A and B mutual supplements of M with $\frac{M}{A}$ and $\frac{M}{B}$ are Hollow-Lifting-Quasi-Discrete Modules then

$$\mathbf{M = A + B}$$

the two submodules A

Proof:

and B of M which are mutual supplements in M, with $\frac{M}{A}$ and $\frac{M}{B}$ are Hollow-Lifting-Quasi-Discrete Modules.

Thus by (proposition 5-1), [12], A and B are coclosed submodules of M, clearly that M is Hollow-Lifting, then, by (proposition 5-5), [13], A and B are direct summands of M since $M = A + B$ and M has (H_3),then $A \cap B$ is adirect summand of M so $M = (A \cap B) + K$, for some $K \leq M$.

Since B is asupplement of A then $A \cap B \ll B$ and hence $A \cap B \ll M$,so M=K and $A \cap B = 0$, Thus we get $M = A + B$.

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