On ϑ-Open Set and Some of its Applications

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Abstract

In this work, we study and introduce new type of open sets is called $\boldsymbol{\vartheta}$-open. We characterize these sets and investigate some of their mainly properties. Further, we present various functions are associated with $\boldsymbol{\vartheta}$-open, called $\boldsymbol{\vartheta}$-open, M $\boldsymbol{\vartheta}$-open, and weakly $\boldsymbol{\vartheta}$-open. We also discuss many characterizations, properties, and relations are discussed. Finally, we study $\boldsymbol{\vartheta}D$-separation axioms by using $\boldsymbol{\vartheta}D$-set

Keywords: $\boldsymbol{\vartheta}$-open set, $\boldsymbol{\vartheta}$closed set, $\boldsymbol{\vartheta}$-open function, M $\boldsymbol{\vartheta}$-open function, weakly $\boldsymbol{\vartheta}$-open function, $\boldsymbol{\vartheta}D$-set

حول المجموعة المفتوحة من النمط ـ ϑ وبعض تطبيقاتها

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الخلاصة

في هذا البحث قمنا بدراسة نوع جديد من المجموعات المفتوحة اسميناها المجموعة المفتوحة من النمط $\boldsymbol{\vartheta}$ حيث قمنا بدراسة عدة تمييزات حول المجموعات المفتوحة من النمط $\boldsymbol{\vartheta}$ وبرهنا عدة نظريات حول هذه المجموعة و عرفنا عدة دوال مرتبطة حول تلك المجموعة منها الدوال المفتوحة و المفتوحة الضعيفة من النمط $\boldsymbol{\vartheta}$ و درسنا العلاقات التي تربط بينها. وقمنا بدراسة بديهيات الفصل من النمط $\boldsymbol{\vartheta}D$ وذلك باستخدام المجموعة من النمط $\boldsymbol{\vartheta}D$.

الكلمات المفتاحية: المجموعة المفتوحة ϑ، المجموعة المغلقة ϑ، الدالة المفتوحة ϑ، الدالة المفتوحة-ϑ M، الدالة المفتوحة الضعيفة ϑ، المجموعة من النمط $\boldsymbol{\vartheta}D$.

Introduction

In 1963, Levine N. [1], detected and discussed the notion of semi-open set furthermore, semi-continuity properties were investigated. The concept of δ-open was introduced by Velicko N. [2], he studied some of their fundamental properties. Since then the notion had been studied by several literatures. Later, Ekici E. [3] discussed e^*-open and$(D,S)^*$.In 2011, Al-magharabi and Mubarki [4] studied Z-open and z-continuous functions. After that Mubarki and others [5] introduced β^*-open set and β^*- continuous functions. In 1985, Rose D. and Jankovich [6], [7] have defined and studied the concepts of weakly open and weakly closed mappings in topological spaces.

Preliminaries

In this work any subset W of a topological space $(X,\Im)$, $\Im int(W)$, $\Im cl(W)$ are denoted for interior and closure respectively.

Definition 2.1: Consider U be any subset of a topological space $(X,\Im)$ is named by semi-open [1] (resp., pre-open [8] , α-open [9], e^*-open [3] and β-open [10]) if $U \subseteq \Im cl\, \Im int\,(U)$ (resp., $U \subseteq \Im int\, \Im cl(U)$, $U \subseteq \Im int\, \Im cl\, \Im int(U)$, $U \subseteq \Im cl\, \Im int\, \Im cl_\delta(U)$and $U \subseteq \Im cl\, \Im int\, \Im cl(U)$).

Definition 2.2: [2] Consider W be any subset of topological space $(X,\Im)$is named by θ-open if any $x \in W$, there is an open set G s.t. $x \in G \subseteq \Im cl(G) \subseteq W$.
The complement θ-open set is called θ-closed.

Definition 2.3: [2] Consider W be any subset of topological space $(X,\Im)$is named δ-open if for each $x \in U$, there exists an open set G such that $x \in G \subseteq \Im int\, \Im cl(G) \subseteq W$.
The complement δ-open set is called δ-closed.

Definition 2.4: The union of any semi-open [1](resp., pre-open [8], α-open [6], θ-open [2], and β-open [10], δ-open [11]) of topological space $(X,\Im)$ sets contained in a subset A is called semi-interior (resp., pre-interior, α-interior, δ-interior, θ-interior, e^*-interior ,and β-

interior) of A. And its denoted by $sint(A)$ (resp., $\Im pint(A)$, $\Im\alpha int(A)$, $\Im int_{\theta}(A)$, $\Im int_{\delta}(A)$, $\Im e^{*}int(A)$ and $\Im\beta int(A)$).

Definition 2.5: The intersection of all semi-closed (resp., pre-closed, α-closed, θ-closed , δ-closed and β-closed) of topological space $(X,\Im)$ containing subset A is called semi-closure, pre-closure, α-closure, θ-closure, and β-closure of A, and its denoted by $\Im scl(A)$, $\Im pcl(A)$, $\Im\alpha cl(A)$, $\Im cl_{\theta}(A)$, $\Im cl_{\delta}(A)$and $\Im\beta cl(A)$).

Definition 2.6: [12] A subset $W$of a topological space $(X,\Im)$is named by b-open set if $W \subseteq \Im cl\,\Im int\,(W) \cup \Im int\,\Im cl\,(W)$. The collection of all b-open sets of X is denoted by $BO(X)$

Definition 2.7: **[13]** A topological space $(X,\Im)$ is named by locally indiscrete if any open subset of X is closed.

Definition 2.8: [14] A topological space $(X,\Im)$ is named by extremally disconnected if the closure of any open subset of topological space X is also open.

Proposition 2.9: [15] Consider W be a subset of topological space$(X,\Im)$. If $W \in \beta O(X)$, then $\Im cl(W) = \Im cl_{\delta}(W)$

Definition 2.10: A mapping $f: (X,\Im) \longrightarrow (Y,\zeta)$ is named by

1) contra closed [16] if $f(U)$ is open set in Y, for every closed set U in X.

2) weakly open [6] if $f(U) \subseteq \zeta int\left(f\left(\Im cl(U)\right)\right)$ for every subset U in X.

Definition 2.11: [17] A map $f: (\mathrm{X},\Im) \longrightarrow (\mathrm{Y},\zeta)$ is named strongly continuous if for each ubset U of X, $f\left(\Im cl(U)\right) \subseteq f(U)$.

ϑ −open set

Definition 3.1.: The subset W of a topological space $(X,\Im)$ is named

1) **ϑ**-open set if$W \subseteq \Im cl\,\Im int\,\,(W) \cup \Im int\,\Im cl\,\Im int\,\Im cl_{\delta}(W)$.

2) **ϑ**-closed set if $\Im int\,\Im cl\,(W) \cap \Im cl\,\Im int\,\Im cl\,\Im int_{\delta}(W) \subseteq W$.

The collection of every $\boldsymbol{\vartheta}$-open sets (resp., $\boldsymbol{\vartheta}$closed) in topological space $(X,\mathfrak{I})$ is denoted by $\boldsymbol{\vartheta}O(X)$ (resp., $\boldsymbol{\vartheta}C(X)$).

Definition 3.2: Let $\mathbb{N}$ be a subset of a topological space $(X,\mathfrak{I})$and let $x \in X$. We called that $\mathbb{N}$ is $\boldsymbol{\vartheta}$-neighborhood of x , if there is $\boldsymbol{\vartheta}$-open set U such that $x \in U \subseteq \mathbb{N}$.

Proposition 3.3: Every α -open is $\boldsymbol{\vartheta}$-open set.

Proof: Assume that W be α –open subset of topological space $(X,\mathfrak{I})$, then $W \subseteq \mathfrak{I}int\, \mathfrak{I}cl\, \mathfrak{I}int\ (W) \subseteq \mathfrak{I}int\, \mathfrak{I}cl\, \mathfrak{I}int\, \mathfrak{I}cl_{\delta}(W)$. Hence A is $\boldsymbol{\vartheta}$-open set in topological space $(X,\mathfrak{I})$.

Proposition 3.4: For any subset A of topological space $(X,\mathfrak{I})$. If A is semi-open set, then A is $\boldsymbol{\vartheta}$-open set.

Proof: Straightforward.

However, the inverse direction of Proposition 3.4 may not satisfy in general as shown in the next example

Example 3.5: Consider $X = \{a,b,c,d\}$ with the topology $\mathfrak{I} = \{\phi,X,\{d\},\{b,c\},\{b,c,d\},\{a,b,c\}\}$. clearly $\{b\}$ is $\boldsymbol{\vartheta}$-open set but it is not semi-open set.

Proposition 3.6: Every $\boldsymbol{\vartheta}$-open is $e^{*} -$ open set.

Proof: Straightforward.

Proposition 3.7: Every β –open and $\boldsymbol{\vartheta}$-open set is b – open set

Proof: Consider W be a $\boldsymbol{\vartheta}$-open set in topological space$(X,\mathfrak{I})$, then $W \subseteq \mathfrak{I}cl\, \mathfrak{I}int\ (W) \cup \mathfrak{I}int\, \mathfrak{I}cl\, \mathfrak{I}int\, \mathfrak{I}cl_{\delta}(W)$. And since A is β –open, then by Proposition 2.9, $W \subseteq \mathfrak{I}cl\, \mathfrak{I}int\ (W) \cup \mathfrak{I}int\, \mathfrak{I}cl(W)$. Hence W is b-open.

Proposition 3.8: For any subset W of topological space $(X,\mathfrak{I})$, if $W \in \delta C(X) \cap \boldsymbol{\vartheta}O(X)$, then $W \in BO(X)$

Proof: straightforward.

Proposition 3.9: Let $\{W_\gamma : \gamma \in I\}$ be a collection of ϑ-open sets subsets of topological space $(X, \Im)$. Then $\cup\{W_\gamma : \gamma \in I\}$ is ϑ-open set.

Proof: Consider W_γ be an ϑ-open set for each γ. Then $W_\gamma \subseteq$ $\Im cl\ \Im int\ (W_\gamma) \cup \Im int\ \Im cl\ \Im int\ \Im cl_\delta(W_\gamma)$. That is $\cup W_\gamma \subseteq$
$\cup(\Im cl\ \Im int\ (W_\gamma) \cup \Im\ int\ \Im cl\ \Im int\ \Im cl_\delta(W_\gamma)) =$
$(\cup \Im cl\ \Im int\ (W_\gamma)) \cup \left(\cup \Im int\ \Im cl\ \Im int\ \Im cl_\delta\ (W_\gamma)\right) \subseteq$
$(\Im cl \cup \Im int\ (w_\gamma)) \cup (\Im int\ \cup \Im cl\ \Im int\ \Im cl_\delta(W_\gamma)) \subseteq$
$\left(\Im cl\ \Im int(\cup W_\gamma)\right) \cup (\Im int\ \Im cl\ \cup \Im int\ \Im cl_\delta(W_\gamma)) \subseteq$
$\left(\Im cl\ \Im int(\cup W_\gamma)\right) \cup (\Im int\ \Im cl\ \Im int\ \cup \Im cl_\delta(W_\gamma)) \subseteq$
$\left(\Im cl\ \Im int(\cup W_\gamma)\right) \cup \left(\Im int\ \Im cl\ \Im int\ \Im cl_\delta(\cup W_\gamma)\right)$. Thus $\cup\{W_\gamma : \gamma \in I\}$ is ϑ-open set.

Remark 3.10: Arbitrary intersection of ϑ-closed is also ϑ-closed.

Proof: By complementation.

The intersection of any two is ϑ-open sets need not be ϑ-open set as showing in the following example

Example 3.11: Consider $X = \{a, b, c, d\}$ with the topology $\Im =$ $\{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, then $A = \{a, b, d\}$ and $B = \{c, d\}$ are both ϑ-open sets but $A \cap B = \{d\}$ is not ϑ-open set

Proposition 3.12: Let A be any an open set in topological space $(X, \Im)$ and B be a ϑ-open set in X, then $A \cap B$ is ϑ-open set in X.

Proof: Assume that B be a ϑ-open set in X, then $B \subseteq$
$\Im cl\ \Im int\ (B) \cup \Im int\ \Im cl\ \Im int\ \Im cl_\delta(B) \Longrightarrow A \cap B \subseteq$
$A \cap (\Im cl\ \Im int\ (B) \cup \Im int\ \Im cl\ \Im int\ \Im cl_\delta\ (B)) =$
$(A \cap \Im cl\ \Im int(B)) \cup (A \cap \Im int\ \Im cl\ \Im int\ \Im cl_\delta\ (B)) \subseteq$
$\left(\Im cl\ (A \cap \Im int(B))\right) \cup (\Im int\ (A) \cap \Im int\ \Im cl\ \Im int\ \Im cl_\delta(B) nt\ tion\ n\ set\ in\ nected\ ed\ 124124124124124$

$\big(\Im cl\ (A \cap \Im int\ (B))\big) \cup \big(\Im int(A \cap \Im cl\ \Im int\ \Im cl_{\delta}\ (B))\big) \subseteq$
$\Im cl\ \Im int\ (A \cap B) \cup \Im int\ \Im cl\big(\ \Im int\ \Im cl_{\delta}(A \cap B)\big) =$
$\Im cl\ \Im int\ (A \cap B) \cup \Im int\ \Im cl\ \Im int\big(A \cap \Im cl_{\delta}(B)\big) \subseteq$
$\Im cl\ \Im int\ (A \cap B) \cup \Im int\ \Im cl\ \Im int\ \Im cl_{\delta}(A \cap B)$. Hence $A \cap B$ is $\boldsymbol{\vartheta}$-open set in X

Proposition 3.13: Let $(Y, \Im_Y)$ be an open subspace of topological space $(X, \Im)$ and let A be any set in Y. If A is $\boldsymbol{\vartheta}$-open set in Y, then A is $\boldsymbol{\vartheta}$-open set in X.

Proof: Let A be $\boldsymbol{\vartheta}$-open set in Y, then $A \subseteq \Im cl_Y\ \Im int_Y\ (A) \cup \Im int_Y\ \Im cl_Y\ \Im int_Y\ \Im cl_{\delta Y}(A)$. Since Y is open, then $A = A \cap \Im int(Y) \subseteq$
$(\Im cl_Y\ \Im int_Y\ (A) \cup \Im int_Y\ \Im cl_Y\ \Im int_Y \Im cl_{\delta Y}(A)) \cap \Im int(Y)$. Therefore $A \subseteq$
$(\Im cl_Y\ \Im int_Y\ (A) \cap \Im int(Y)) \cup (\Im int_Y\ \Im cl_Y\ \Im int_Y\ \Im cl_Y(A) \cap \Im int(Y))$. Now,
$\Im cl_Y\ \Im int_Y(A) \cap \Im int(Y) = (\Im cl\ \Im int_Y\ (A) \cap Y) \cap \Im int(Y) =$
$(\Im cl \Im\ int_Y\ (A) \cap \Im int(Y)) \cap Y = \big(\Im cl\ \Im int_Y(A)\big) \cap Y \subseteq \Im cl(\Im int_Y\ (A) \cap Y) =$
$\Im cl(\Im int_Y\ (A) \cap \Im int(Y)) = \Im cl\ \Im int\ (A)$
Also, $\Im int_Y\ \Im cl_Y\ \Im int_Y\ \Im cl_{\delta Y}(A) \cap \Im int(Y) = \Im int\ \Im cl_Y\ \Im int_Y\ \Im cl_{\delta Y}(A) =$
$\Im int\ \Im cl\big(\Im int_Y\ \Im cl_{\delta Y}(A)\big) \cap Y \subseteq \Im int\ \Im cl(\Im int_Y\ \Im cl_{\delta Y}(A) \cap Y) =$
$\Im int\ \Im cl(\Im int_Y\ \Im cl_{\delta Y}(A) \cap \Im int(Y)) = \Im int\ \Im cl\ \Im int\ \Im cl_{\delta Y}(A) =$
$\Im int\ \Im cl\ \Im int\ (\Im cl_{\delta Y}(A) \cap Y) \subseteq \Im int\ \Im cl\ \Im int\ \Im cl_{\delta}(A)$. Therefore $A \subseteq$
$\Im cl\ \Im int(A) \cup \Im int\ \Im cl\ \Im int\ \Im cl_{\delta}(A)$. Hence A is $\boldsymbol{\vartheta}$-open set in X.

Remark 3.14: Let $(Y, \Im_Y)$ be any subspace of topological space $(X, \Im)$ and let A be any set in Y. If A is $\boldsymbol{\vartheta}$-open set in X, then A is $\boldsymbol{\vartheta}$-open set in Y

Proof: Straightforward.

Definition 3.15: Let $(X, \Im)$ be any topological space and A be a subset of X . A point p of subset U of X is called $\boldsymbol{\vartheta}$- interior point of A, if there exists $\boldsymbol{\vartheta}$- open set G such that $p \in G \subseteq U$. The set of every $\boldsymbol{\vartheta}$- interior points of A is said to be $\boldsymbol{\vartheta}$- interior set and its denoted by $\Im int_{\vartheta}(U)$.

Proposition 3.16: For any subset U of topological space (X,τ), $\Im int(U) \subseteq \Im\alpha int(U) \subseteq \Im sint(U) \subseteq \Im int_{\vartheta}(U) \subseteq \Im e^{*}int(U)$.

Proof: Straightforward.

Proposition 3.17: If A and B are sets in topological space (X,τ), then

1)$\Im int_{\vartheta}(\phi) = \phi$ and $\Im int_{\vartheta}(X) = X$

2) $\Im int_{\vartheta}(U) \subseteq U$

3) If $U \subseteq V$ then $\Im int_{\vartheta}(U) \subseteq \Im int_{\vartheta}(V)$

Definition 3.18: Let (X,τ) be any topological space and A be a subset of X. The intersection of all $\boldsymbol{\vartheta}$- closed sets containing A is called $\boldsymbol{\vartheta}$- closure of U and is denoted by $\boldsymbol{\Im} cl_{\vartheta}(U)$

Proposition 3.19: Let G be any subset of a topological space(X,τ). Then $x \in \boldsymbol{\Im} cl_{\vartheta}(G)$ iff for every $\boldsymbol{\vartheta}$-open set U containing x, $U \cap G \neq \phi$.

Proof: Straightforward.

Proposition 3.20: For any subset U of topological space (X,τ), $\boldsymbol{\Im} cl_{\vartheta}(U) \subseteq \Im scl(U) \subseteq \Im cl(U)$.

Proof. Obvious.

Some $\boldsymbol{\vartheta}$-open mappings:

Definition 4.1: A map $f:(X,\Im) \longrightarrow (Y,\zeta)$ is named

1) M $\boldsymbol{\vartheta}$-open if the image of any $\boldsymbol{\vartheta}$-open set in X is $\boldsymbol{\vartheta}$-open subset of Y.

2) M $\boldsymbol{\vartheta}$-closed if the image of any $\boldsymbol{\vartheta}$-closed set in X is $\boldsymbol{\vartheta}$-closed subset of Y.

Definition 4.2: A function $f:(X,\Im) \longrightarrow (Y,\zeta)$ is named

1) $\boldsymbol{\vartheta}$- open if the image of every open set in X is an $\boldsymbol{\vartheta}$-open subset of Y.

2) pre $\boldsymbol{\vartheta}$-open if the image of every $\boldsymbol{\vartheta}$-open set in X is an open subset of Y.

Proposition 4.3.: Let $f:(X,\Im) \longrightarrow (Y,\zeta)$be a function, then the following are equivalents:

1) f is M $\boldsymbol{\vartheta}$-open

2) For every subset G of X, $\mathrm{f}(\Im\mathrm{int}_\vartheta(\mathrm{G})) \subseteq \zeta\mathrm{int}_\vartheta(\mathrm{f}(\mathrm{G}))$.

3) For every $x \in X$, and for each $\boldsymbol{\vartheta}$-neighborhood W of x in X, there exists $\boldsymbol{\vartheta}$-neighborhood H of $f(x)$ in Y such that $H \subseteq f(W)$.

Proof: (1)⟹(2) Assume that f is M $\boldsymbol{\vartheta}$-open. Since $\Im int_\vartheta(G) \subseteq G$, then $f(\Im int_\vartheta(G)) \subseteq f(G)$. By definition of M $\boldsymbol{\vartheta}$-open, $f(\Im int_\vartheta(G))$ is $\boldsymbol{\vartheta}$-open set in Y contained in $f(G)$. Thus $f(\Im int_\vartheta(A)) \subseteq \zeta int_\vartheta(f(G))$.

(2) ⟹(3) Let U be $\boldsymbol{\vartheta}$-neighborhood of x, then there is a $\boldsymbol{\vartheta}$-open set V in X such that $x \in W \subseteq U$. By (2), we get $f(W) = f(\Im int_\vartheta(W)) \subseteq \zeta int_\vartheta(f(W))$. that is $H = f(W)$ be a $\boldsymbol{\vartheta}$-open set in Y s.t. $f(x) \in H \subseteq f(W)$.

(3) ⟹(1) Consider U be an $\boldsymbol{\vartheta}$-open set in X then for any $x \in U$, there exits $\boldsymbol{\vartheta}$-neighborhood W of $f(x)$ such that $W_{f(x)} \subseteq f(U)$. This implies that $f(U) = \cup\{W_{f(x)}: x \in U\}$ is $\boldsymbol{\vartheta}$-open set. Hence f is M $\boldsymbol{\vartheta}$-open.

Proposition 4.4: Let $f:(X,\Im) \longrightarrow (Y,\zeta)$be a surjective function, then f is M $\boldsymbol{\vartheta}$-open iff the image of every $\boldsymbol{\vartheta}$-closed set in X is $\boldsymbol{\vartheta}$-closed set in Y.

Proof: Obvious.

Proposition 4.5: Let $f:(X,\Im) \longrightarrow (Y,\zeta)$ be a map and β be any base for topological $(X,\Im)$. Then f **is** $\boldsymbol{\vartheta}$-open if and only if $f(U)$ is $\boldsymbol{\vartheta}$-open set for each $U \in \beta$

Proof: Assume that f is ϑ-open and since $U \in \beta$, then B is an open set in topological space $(X,\Im)$and so $f(U)$ is ϑ-open set in (Y,ζ).

Conversely, let A be an open set, then $A = \bigcup_i U_i$ for $B_i \in \beta$. It follows that $f(A) = f(\bigcup_i U_i) = \bigcup_{\boldsymbol{i}} f(U_i)$. By hypothesis, $f(B_i)$ is ϑ-open and by Proposition 3.9, $f(A)$ is ϑ-open. Hence f is ϑ-open.

Proposition 4.6: A surjective function $f: X \longrightarrow Y$ is pre-$\boldsymbol{\vartheta}$-open if and only if $f(G) \setminus f(X \setminus G)$ is an open set in Y whenever A is $\boldsymbol{\vartheta}$-open set in X.

Proof: Suppose that f is pre-$\boldsymbol{\vartheta}$-open and let G be $\boldsymbol{\vartheta}$-open set, so $f(G)$ is an open set in Y. Now $f(G) \setminus f(X \setminus G) = f(G) \cap [Y \setminus f(X \setminus G)]$, since $Y \setminus f(X \setminus G)$ is an open set, therefore $f(G) \setminus f(X \setminus G)$ is an open set in Y.

Conversely, suppose that for $\boldsymbol{\vartheta}$-open set A in X, $f(G) \setminus f(X \setminus G)$ is an open set in Y. Let B be an $\boldsymbol{\vartheta}$-open set in X, then $f(B) = Y \setminus f(X \setminus B) \setminus f(B)$ is an open set in Y. Hence f is pre-$\boldsymbol{\vartheta}$-open.

Proposition 4.7: Let $f: (X, \mathfrak{I}) \longrightarrow (Y, \zeta)$ be a function then f is M$\boldsymbol{\vartheta}$-open if and only if $\mathfrak{I}int_{\vartheta}\big(f^{-1}(V)\big) \subseteq f^{-1}\big(\zeta int_{\vartheta}(V)\big)$ for any $V \subseteq Y$.

Proof: Suppose that f is M $\boldsymbol{\vartheta}$-open. Let A be arbitrary subset of Y, then $f^{-1}(V)$ is a subset of X. By theorem 4.3(2), $f\left(\mathfrak{I}int_{\vartheta}\big(f^{-1}(V)\big)\right) \subseteq \zeta int_{\vartheta}\left(f\big(f^{-1}(V)\big)\right)$ this implies that $f\left(\mathfrak{I}int_{\vartheta}\big(f^{-1}(V)\big)\right) \subseteq \zeta int_{\vartheta}(V)$. Therefore, $\mathfrak{I}int_{\vartheta}\big(f^{-1}(V)\big) \subseteq f^{-1}\big(\zeta int_{\vartheta}(V)\big)$ for $V \subseteq Y$.

Conversely, suppose the hypothesis is satisfied and let W be a $\boldsymbol{\vartheta}$-open set in X, then $f(W)$ is a subset of Y. By hypothesis $\mathfrak{I}int_{\vartheta}\left(f^{-1}\big(f(W)\big)\right) \subseteq f^{-1}\left(\zeta int_{\vartheta}\big(f(W)\big)\right)$ that is $\mathfrak{I}int_{\vartheta}(W) \subseteq f^{-1}\left(\zeta int_{\vartheta}\big(f(W)\big)\right)$. Consequently, $f(W) \subseteq \zeta int_{\vartheta}\big(f(W)\big)$. Therefore $f(W)$ is $\boldsymbol{\vartheta}$-open. Hence f is M $\boldsymbol{\vartheta}$-open.

Proposition 4.8: Let $f: (X, \tau) \longrightarrow (Y, \zeta)$and $g: (Y, \zeta) \longrightarrow (Z, \rho)$ be two functions then

1) If f and g are both M $\boldsymbol{\vartheta}$-open, then $g \circ f: (X, \tau) \longrightarrow (Z, \rho)$ is also M $\boldsymbol{\vartheta}$-open

2) If f is pre $\boldsymbol{\vartheta}$open and g is M $\boldsymbol{\vartheta}$-open, then $g \circ f: (X, \tau) \longrightarrow (Z, \rho)$ is pre $\boldsymbol{\vartheta}$-open

Proof: Straightforward.

Proposition 4.9: Let $f: (X, \tau) \longrightarrow (Y, \zeta)$ and $g: (Y, \zeta) \longrightarrow (Z, \gamma)$ be two functions. if f is surjective and continuous function, and $g \circ f: (X, \tau) \longrightarrow (Z, \gamma)$ is $\boldsymbol{\vartheta}$-open, then g is $\boldsymbol{\vartheta}$-open

Proof: Let A be an open subset of (Y, ζ). Since f is continuous, then $f^{-1}(A)$ is an open set in X. But $g \circ f$ is $\boldsymbol{\vartheta}$-open, thus $g \circ f\left(f^{-1}(A)\right) = g(A)$ is $\boldsymbol{\vartheta}$-open set. Hence g is $\boldsymbol{\vartheta}$-open.

Definition 4.10: A map $f: (X, \Im) \longrightarrow (Y, \zeta)$ is named by weakly $\boldsymbol{\vartheta}$-open if $f(A) \subseteq \zeta int_{\vartheta}\left(f\left(\Im cl(A)\right)\right)$, for every A is an open subset of X.

Definition 4.11: A function $f: (X, \Im) \longrightarrow (Y, \zeta)$ is named by weakly $\boldsymbol{\vartheta}$-closed if $\zeta cl_{\vartheta}\left(f\left(\Im int(B)\right)\right) \subseteq f(B)$, for every B is a closed subset of X.

It is clear that every weakly open is weakly $\boldsymbol{\vartheta}$-open.

Theorem 4.12: Let X be locally indiscrete space, then $f: (X, \Im) \longrightarrow (Y, \zeta)$ is weakly $\boldsymbol{\vartheta}$-open iff it is $\boldsymbol{\vartheta}$-open.

Proof: Sufficiently, let A be an open set in locally indiscrete space X. Since f is weakly $\boldsymbol{\vartheta}$-open, then $f(A) \subseteq \zeta int_{\vartheta}\left(f\left(\Im cl(A)\right)\right) = \zeta int_{\vartheta}\left(f(A)\right)$ and so $f(A)$ is $\boldsymbol{\vartheta}$-open set in Y. Hence f is $\boldsymbol{\vartheta}$-open.

Necessity, let B be an open set in space X. Since f is $\boldsymbol{\vartheta}$-open, then $f(B) = \zeta int_{\vartheta}\left(f(B)\right) \subseteq \zeta int_{\vartheta}\left(f\left(\Im cl(B)\right)\right)$. Hence f is weakly $\boldsymbol{\vartheta}$-open.

Proposition 4.13: If $f: (X, \Im) \longrightarrow (Y, \zeta)$ is weakly $\boldsymbol{\vartheta}$-open with strongly continuous, then it is $\boldsymbol{\vartheta}$-open.

Proof: Assume that A be an open set in space X. Since f is weakly $\boldsymbol{\vartheta}$-open, then $f(A) \subseteq \zeta int_{\vartheta}\left(f\left(\Im cl(A)\right)\right)$ .but f is strongly continuous, thus $f(A) \subseteq \zeta int_{\vartheta}\left(f\left(\Im cl(A)\right)\right) \subseteq \zeta int_{\vartheta}\left(f(A)\right)$. Therefore $f(A)$ is $\boldsymbol{\vartheta}$-open set in Y. Hence f is $\boldsymbol{\vartheta}$-open.

Proposition 4.14: Every contra closed is weakly $\boldsymbol{\vartheta}$-open.

Proof: Let $f:(X,\Im) \longrightarrow (Y,\zeta)$ be contra closed and let A be an open set in space X, then $f(A) \subseteq f(\Im cl(A))$ .Also , f is contra closed $f(\Im cl(A)) = \Im int\left(f(\Im cl(A))\right) \subseteq \zeta int_{\vartheta}\left(f(\Im cl(A))\right)$. Hence f is weakly $\boldsymbol{\vartheta}$-open.

Proposition 4.15: A function $f:(X,\Im) \longrightarrow (Y,\zeta)$ is weakly $\boldsymbol{\vartheta}$-open if and only if for any $x \in X$, and every open set U of X s.t. $x \in U$, there exists a $\boldsymbol{\vartheta}$-open set V in Y containing $f(x)$ such that $V \subseteq f(\Im cl(U))$.

Proof: Sufficiently, let U be an open set in X containing x. Since f is weakly $\boldsymbol{\vartheta}$-open, then $f(U) \subseteq \zeta int_{\vartheta}\left(f(\Im cl(U))\right)$. Set $V = \zeta int_{\vartheta}\left(f(\Im cl(U))\right)$ is a $\boldsymbol{\vartheta}$-open set in Y containing $f(x)$ such that $V \subseteq f(\Im cl(U))$.

Necessity, let U be an open set in X. Now, for each $x \in U$, there exists $\boldsymbol{\vartheta}$-open set V in Y containing $f(x)$ such that $V \subseteq f(\Im cl(U))$ and so, $f(U) \subseteq V \subseteq f(\Im cl(U))$ and since V is$\boldsymbol{\vartheta}$-open set, then $V \subseteq \zeta int_{\vartheta}\left(f(\Im cl(U))\right)$. Therefore, $f(U) \subseteq \zeta int_{\vartheta}\left(f(\Im cl(U))\right)$. Hence f is weakly $\boldsymbol{\vartheta}$-open.

Theorem 4.16: For a function $f:(X,\Im) \longrightarrow (Y,\zeta)$, the following are equivalents:

1) f is weakly $\boldsymbol{\vartheta}$-open

2) $f(\Im int(B)) \subseteq \zeta int_{\vartheta}(f(B))$, for each closed set B of X

3) $f(\Im int\, \Im cl\,(A)) \subseteq \zeta int_{\vartheta}\left(f(\Im cl(A))\right)$, for each open set A of X.

Proof: (1)⟹(2) let B be closed in X, then $\Im int(B)$ is an open set in X. By (1), $f(\Im int(B)) \subseteq \zeta int_{\vartheta}\left(f(\Im cl\, \Im int(B))\right)$ and since B is closed, then it is pre-closed and so $\zeta int_{\vartheta}\left(f(\Im cl\, \Im int(B))\right) \subseteq \zeta int_{\vartheta}(f(B))$ that is $f(\Im int(B)) \subseteq \zeta int_{\vartheta}(f(B))$.

(2) ⟹(3) Let A be an open set in X, then $cl(A)$ is closed set in X. By applying(2), we have $f(\Im int\Im\, cl\,(A)) \subseteq \zeta int_\vartheta \left(f(\Im cl(A))\right)$.

(3) ⟹(1) let U be an open set in X, then U is pre-open and by (3), we get $f(U) \subseteq f(\Im int\, \Im cl\,(U)) \subseteq \zeta int_\vartheta \left(f(\Im cl(A))\right)$. Hence f is weakly $\boldsymbol{\vartheta}$-open.

Proposition 4.17: Let $f:(X,\tau) \longrightarrow (Y,\zeta)$ be bijective function then f is weakly $\boldsymbol{\vartheta}$-open if and only if $f(\Im int_\theta(B)) \subseteq \zeta int_\vartheta(f(B))$ for any subset B of X

Proof: Sufficiently, let B be subset of a space X and $y \in f(\Im int_\theta(B))$, then there exists $x \in \Im int_\theta(B)$ and so there exists an open set G such that $x \in G \subseteq \Im cl(G) \subseteq B$ therefore, $y = f(x) \in f(G) \subseteq f(\Im cl(G)) \subseteq f(B)$. Since f is weakly $\boldsymbol{\vartheta}$-open, then $y \in f(G) \subseteq \zeta int_\vartheta \left(f(\Im cl(G))\right) \subseteq \zeta int_\vartheta(f(B))$. Hence $f(\Im int_\theta(B)) \subseteq \zeta int_\vartheta(f(B))$.

Necessity, let U be an open subset of a space X. Since $U \subseteq \Im int_\theta(\Im cl(U))$, then $f(U) \subseteq f(\Im int_\theta\, \Im cl\,(U)) \subseteq \zeta int_\vartheta \left(f(\Im cl(U))\right)$. Hence f is weakly $\boldsymbol{\vartheta}$-open.

Proposition 4.18: If $f:(X,\Im) \longrightarrow (Y,\zeta)$ is weakly $\boldsymbol{\vartheta}$-open and strongly continuous, then the image of every open set in X, is e^*-open set in Y.

Proof: Let A be an open set in space X. Since f is weakly $\boldsymbol{\vartheta}$-open, then $f(A) \subseteq \zeta int_\vartheta \left(f(\Im cl(A))\right)$ and since f is strongly continuous, then $f(A) \subseteq \zeta int_\vartheta \left(f(\Im cl(A))\right) \subseteq \zeta int_\vartheta(f(A)) \subseteq \zeta\beta int(f(A))$. Therefore $f(A)$ is e^*-open set in Y.

Proposition 4.19: If $f:(X,\Im) \longrightarrow (Y,\zeta)$ is almost open and closed, then it is weakly $\boldsymbol{\vartheta}$-open.

Proof: Let A be an open set in space X. Since f is almost open , then $f(A) \subseteq \zeta int\, \zeta cl\,(f(A))$ and since f is closed, then $f(A) \subseteq \zeta int\, \zeta cl\,(f(A)) \subseteq \zeta int \left(f(\Im cl(A))\right) \subseteq \zeta int_\vartheta \left(f(\Im cl(A))\right)$. Hence f is weakly $\boldsymbol{\vartheta}$-open.

ϑD-set

Definition 5.1: A subset A of topological space $(X,\Im)$is named **ϑD**-set if there exist two **ϑ**-open sets U and V such that $U \neq X$ and $A = U - V$.

Proposition 5.2: Every proper **ϑ**-open set is **ϑD**-set.

Proof: Let W be proper subset of topological space $(X,\Im)$ and since $W = W - \phi$, then W is **ϑD**-set.

However, the converse is not true in general as showing in the next example.

Example 5.3: Consider $X = \{a,b,c,d\}$ with the topology $\Im = \{\phi, X, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$, then $G = \{a,c\}$ is **ϑD**-set but it is not **ϑ**-open set.

Definition 5.4: A topological space $(X,\Im)$ is named

1) **ϑ**T_0 −space if for each two distinct points a and b of X, there is a **ϑ**-open Wcontaining a but not b or containing b but not a.

2) **ϑ**T_1 −space if for each two distinct points a and b of X, there are **ϑ**-open sets U and V s.t. $a \in U, b \notin U, b \in V$, and $a \notin V$.

3) **ϑ**T_2 −space if for each two distinct points a and b of X, there are **ϑ**-open sets U and V s.t. $a \in U$, $b \in V$ and $U \cap V = \phi$.

Definition 5.5: A topological space $(X,\Im)$ is named to be

1) **ϑ**D_0 −space if for each two different points a and b of X, there is a **ϑD**-open containing a but not b or containing b but not a.

2) **ϑ**D_1 −space if for each two different points a and b of X, there are **ϑ**D-open sets U and W s.t. $a \in U, b \notin U, b \in W$, and $a \notin W$.

3) **ϑ**D_2 −space if for each two different points a and b of X, there is **ϑ**D-open sets U and W s.t. $a \in U$, $b \in W$ and $U \cap W = \phi$.

Remark 5.6:

1) Every ϑT_i −space is ϑT_{i-1} −space. For $i = 1,2$

2) Every $\boldsymbol{\vartheta D_i}$ −space is $\boldsymbol{\vartheta D_{i-1}}$ −space. For $i = 1,2$

Proposition 5.7: Every $\boldsymbol{\vartheta D_1}$ −space is ϑT_0 −space.

Proof: Let a and b are two distinct points in $\boldsymbol{\vartheta D_1}$ −space$(X,\mathfrak{I})$, then there exist two $\boldsymbol{\vartheta}$D- sets U and V such that $a \in U = K - L$, $b \notin U = K - L$, $b \in V = M - N$, $a \notin V = M - N$, and $K, M \neq X$. When $a \notin V$, there are two options

1) $a \notin M$, and since $b \in V$, then $b \in M$, $M$is $\boldsymbol{\vartheta}$-open set

2) $a \in M$ and $a \in N$. But $b \in V = M - N$, thus $b \notin N$, N is $\boldsymbol{\vartheta}$-open set. Hence $(X,\mathfrak{I})$ is ϑT_0 −space.

Proposition 5.8: Every ϑT_i −space is $\boldsymbol{\vartheta D_i}$ −space for $i = 0,1,2$.

Proof: when $i = 1$, let a and b be two different points in ϑT_1 −space(X,τ), then there exist two $\boldsymbol{\vartheta}$-open sets G and H such that $a \in G, b \notin G, b \in H$, and $a \notin H$. It follows that $a \in G - H$, $b \notin G - H$, $b \in H - G$, and $a \notin H - G$ where $G, H \neq X$. Hence $(X,\tau)$is $\boldsymbol{\vartheta D_1}$ −space.

Proposition 5.9: Let $(X,\mathfrak{I})$ be a topological space then the following are equivalents:

1) X is ϑD_2 −space.

2) X is ϑD_1 −space.

Proof: 1⟹2 By Remark 5.6

2⟹1 Let a and b be two different points in ϑD_1 −space X , then there exist two $\boldsymbol{\vartheta D}$-sets U and V such that $a \in U = G_1 - G_2$, $b \notin U = G_1 - G_2$, $b \in V = G_3 - G_4$, $a \notin V = G_3 - G_4$. For $b \notin U = G_1 - G_2$, we have two issues. isuue1: if $b \notin G_1$, and $b \in G_3 - G_4$, then $b \in G_3 - (G_1 \cup G_4)$. Also, $a \in G_1 - G_2$ and since $a \notin G_3$, then $a \in G_1 - (G_2 \cup G_3)$ with $[G_3 - (G_1 \cup G_4)] \cap [G_1 - (G_2 \cup G_3)] = \phi$.

If $a \in G_3$ and $a \in G_4$and since $b \in G_3 - G_4$, then $G_4 \cap (G_3 - G_4) = \phi$

issue2: $b \in G_1$and $b \in G_2$ and since $a \in G_1$, then $G_1 \cap (G_2 - G_1) = \phi$

Definition 5.10: A topological space $(X, \Im)$ is said to be $\boldsymbol{\vartheta}$-symmetric , if for every $a, b \in X$, $a \in \Im cl_{\vartheta}(\{b\})$ implies that $b \in \Im cl_{\vartheta}(\{a\})$.

Proposition 5.11: For $\boldsymbol{\vartheta}$-symmetric space $(X, \Im)$, then the following are valent:

1) X is $\boldsymbol{\vartheta}T_0$ −space **2)** X is $\boldsymbol{\vartheta}T_1$ −space **3)** X is $\boldsymbol{\vartheta}D_1$ −space

Proof: (1)⇒(2) let a and b are two distinct points in $\boldsymbol{\vartheta}T_0$ −space X, then there exists $\boldsymbol{\vartheta}$-open set Usuch that $a \in U \subseteq X - \{b\}$. It follows $a \notin \Im cl_{\vartheta}(\{b\})$ and since X is $\boldsymbol{\vartheta}$-symmetric space, then $b \notin \Im cl_{\vartheta}(\{a\})$ and so $b \in X - \Im cl_{\vartheta}(\{a\})$.

(2) ⟹ (3) By Remark 5.6

(3) ⟹(1) By Proposition 5.7

Proposition 5.12: Let $f: (X, \tau) \longrightarrow (Y, \zeta)$ be one to one and onto function. If A is $\boldsymbol{\vartheta D}$-set in X, then $f(A)$ is also $\boldsymbol{\vartheta D}$-set in Y.

Proof: Straightforward.

Theorem 5.13: if $f: (X, \Im) \longrightarrow (Y, \zeta)$ is one to one, onto, and M $\boldsymbol{\vartheta}$-open function and $(X, \Im)$is $\boldsymbol{\vartheta}T_i$ −space, then (Y, ζ) is $\boldsymbol{\vartheta}D_i$ −space ($i = 0,1,2$).

Proof: We will prove when $i = 1$, and similarly for others

Let y_1 and y_2 are two different points in $\boldsymbol{\vartheta}T_1$ −space, then there exists x_1 and x_2 such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. But X is $\boldsymbol{\vartheta}T_1$ −space, therefore there exist two $\boldsymbol{\vartheta}$-open sets U and V such that $x_1 \in U$, $x_2 \notin U$, $x_2 \in V$,and $x_1 \notin V$. By Proposition 5.2 and since f is M $\boldsymbol{\vartheta}$-open, then $f(U)$ and $f(V)$ are $\boldsymbol{\vartheta D}$-sets such that $y_1 \in f(U)$, $y_2 \notin f(U)$, $y_2 \in f(V)$and $y_1 \notin f(V)$. Hence Y is $\boldsymbol{\vartheta}D_1$ −space.

Proposition 5.14: let f be one to one and M $\boldsymbol{\vartheta}$-open from $(X, \mathfrak{I})$ onto $\boldsymbol{\vartheta}$-symmetric space(Y, ζ). If $(X, \mathfrak{I})$is $\boldsymbol{\vartheta}T_0$ $-$space, then (Y, ζ) is $\boldsymbol{\vartheta}D_1$ $-$space.

Proof: By Theorem 5.13, and Proposition 5.11.

References

1. Levine N. American Math. Monthly 1963, 70(1), 36-41.
2. Velico N.K., *American. Math. Soc.Trans.* 1968, 78(2), 103-118.
3. Ekici E. Mathematica Moravica 2009, 13(1), 29-36.
4. EL-Magharabi, A. I.; Mubarki, A.M. International Journal of Mathematical Archive EISSN 2011, 2(10), 1819-1827.
5. Mubarki, A. M., Al-Rshudi, M. M. and Al-Juhani, M. A. Journal of Taibah University for science 2014, 8, 142-148.
6. Rose, D. A.; Janković, D.S. Mathematische Nachrichten 1987, 130(1),105-110
7. Janković, D. Acta Mathematica Hungarica 1985, 46(1-2), 83-92
8. El-Deeb S. N.; Mashhour A. S.; Abd El-Monsef M.E *Proc. Math. Phys. Soc.* Egypt 1982, 53, 47-53.
9. Njastad O. *Pacific J. Math.*,1965, 15(3), 961-970.
10. Abd El-Monsef M. E.; El-Deeb S.N.; Mahmoud R.A. Bull. Fac. Sci. Assuit. Univ.1983, (21)1,1-18.
11. Rose D.A. International Journal of Mathematics and Mathematical Sciences 1984, 7(1), 35-40.
12. Andrijevic, D. Matema. Bech. Vesnisk1996, 48, 59-64.
13. Dontchev J., Survey on pre-open sets, The Proceedings of the yatsushiro topological conference 1998, 1-18.
14. Di Maio G.; Noiri T. Indian J. Pure Appl. Math. 1985, 18(3), 226-233.
15. Guo T. Y. J. Central China Normal Univ. Nature Sci. 1981, 21(2), 169-170.
16. Baker, C. W. Math. Today. (Ahmed-abad), 1997,15, 19-24.
17. Levine N. American. Math. Monthly1960, 67, 269.