(p, q) - Fuzzy α^m-Closed Sets in Double Fuzzy Topological Spaces

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Abstract

In this submitted article, we defined new notions of closed sets and called it (p, q)-fuzzy α^m-closed sets in double fuzzy topological spaces. Also, we investigate some characterizations and properties of the sets mentioned above. As a result, we discussed some new and more general relationship between (p, q)-fuzzy α^m-closed set and it's generalized which is called an (p, q)-generalized fuzzy α^m-closed sets which was obtained and compared.

Keywords: Double fuzzy topology; (p, q)-fuzzy α^m-closed set; (p, q)-fuzzy α^m-open set; (p, q)- generalized fuzzy α^m-closed set; (p, q)-generalized fuzzy α^m-open set.

المجموعات المغلقة (p,q) -الفازية- α^m في الفضاءات التبولوجية الفازية المزدوجة

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الخلاصة

في هذه الدراسة قدمنا ودرسنا مصطلح جديد للمجموعات المغلقة تدعى مجموعة المغلقة (p, q)-االفازية-α^m في الفضاءات التبولوجية الفازية المزدوجة حيث تحرينا عن بعض من مميزات وخواص مجموعات المذكورة آنفا. ونتيجة لذلك ناقشنا العديد من العلاقات الجديدة والعامة بين مجموعة المغلقة (p, q)-الفازية- α^m وتعميمها التي تدعى المجموعة المغلقة (p, q)-الفازية المعممة- α^m في الفضاءات التبولوجية الفازية المزدوجة والتي تم التحقق منها ومقارنتها.

الكلمات المفتاحية: التبولوجي الفازي المزدوج، مجموعة المغلقة (p, q)-االفازية-α^m، مجموعة المفتوحة (p, q)-الفازية-α^m، مجموعة المغلقة (p, q)-الفازية المعممة-α^m، مجموعة المفتوحة (p, q)-الفازية المعممة-α^m.

Introduction

The first use of the term of closed sets was study by Levine's [1] in topological space. After that Chang [2] introduced fuzzy topological space. Later, as an extension of Zadeh's study of fuzzy sets [3], Coker [4] defined the topology of intuitionistic fuzzy sets, which the concept of intuitionistic fuzzy sets was introduced by Atanassov [5]. This concept in 2005 terminated by Garcia and Rodabaugh [6], when they suggested that the double fuzzy set is a more appropriate name than intuitionistic and completed that their research project under the name (" double" rather than intuitionistic). The generalization of closed set reported by Abbas 2006 [7] and α-closed by [8]. The goal of this article is to continue and to the allocation study of Fatimah *et al.* [9,10]. Also, we will generalization the study of some properties of α^m-closed sets in topological spaces by Milby [11], then we investigate the behavior of (p, q)-fuzzy α^m-closed set and its various characterization, after that we introduce the concept of (p, q)-generalized fuzzy α^m-closed set and it's complement (p, q)-generalized fuzzy α^m-open. Furthermore, we establish some of their characteristic and properties with various examples.

Preliminaries

Firstly, we will remained the definitions of the most essential concepts defined in double fuzzy topological spaces. Suppose X be any non-empty set and I be the closed unit interval [0,1], $\mathrm{I}p_0 = (0,1]$, $\mathrm{I}q_1 = [0,1)$, the set of all fuzzy subsets on X is denoted by I^X. $p_{t(z)}$ *is* the family of all fuzzy point in X. A fuzzy set λ is called an (p, q)-fuzzy open ((p, q) fo ,for short) if $\tau(\lambda) \geq p$ and $\tau^*(\lambda) \leq q$. Whenever, $p \epsilon \mathrm{I}_{p0}$ and $q \in \mathrm{I}_{q1}$. A fuzzy set λ is called an (p, q)-fuzzy closed ((p, q)-fc, for short) set whenever $\tau(1-\lambda) \geq p$ and $\tau^*(1-\lambda) \leq q$,when $1-\lambda$ is the complement of λ. For any fuzzy sets λ and μ in X, we write $\lambda q \mu$ to mean that λ *is quasiconcident with* μ, that is there exists at least one point $z \in X$ such that $\lambda(z) + \mu(z) > 1$, Negation of such statement is denoted by $\lambda\ \bar{q}\ \mu$ *such that* $\lambda(z) + \mu(z) \leq 1$, whenever fuzzy point in which is special fuzzy set with membership function defined by $z_{t0}(y)$ is equal to t_0, where y = z and equal to 0 where z different of y for each z in X and $t_0 \in I_{p0}$.

Now, we will recall to some of the basic definitions which was already defined by various authors.

Definition 2.1[6, 12] ."A double fuzzy topology space (τ,τ^*) on a non-empty set X is a pair of functions $\tau,\tau^*: I^x \to I$, which satisfies the following properties:

$(O1)$ $\tau(\lambda) \leq 1-\tau^*(\lambda)$ for each $\lambda \in I^x$.

$(O2)$ $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leq \tau^*(\lambda_1) \vee \tau^*(\lambda_2)$ for each $\lambda_1, \lambda_2 \in I^x$.

$(O3)$ $\tau(\vee_{i\in\Gamma} \lambda_i) \geq \wedge_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^*(\vee_{i\in\Gamma} \lambda_i) \leq \vee_{i\in\Gamma} \tau^*(\lambda_i)$ for each $\lambda_i \in I^x, i \in \Gamma$."

The triplex (X,τ,τ^*) is called a double fuzzy topological spaces (dfts, for short).

Definition 2.2 [5, 6]. "If (X,τ,τ^*) be a dfts. Then double fuzzy closure operator and double fuzzy interior operator of $\lambda \in I^X$ are defined by:

$$C_{\tau,\tau^*}(\lambda, p, q) = \wedge \{\mu \in I^x,\ \lambda \leq \mu,\ \tau(1-\mu) \geq p,\ \tau^*(1-\mu) \leq q\},$$

$$I_{\tau,\tau^*}(\lambda, p, q) = \vee \{\mu \in I^x,\ \mu \leq \lambda, \tau(\mu) \geq p, \tau^*(\mu) \leq q\}.$$

where $p \epsilon I_{p0}$ *and* $q \in I_{q1}$ with $p+q \leq 1$."

Definition 2.3. Suppose $(X, \tau, \tau*)$ be a dfts. For each $\lambda, \mu \in I^X$, $p \epsilon I_{p0}$ *and* $q \in I_{q1}$.

1. λ is an (p, q)-fuzzy pre-open set (see [15]) (briefly, (p, q)-fp-open), if $\lambda \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q)$. λ is an (p, q)-fuzzy pre-closed set (briefly, (p, q)-fp-closed), if $C_{\tau,\tau^*}(I_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \lambda$.
2. λ is an (p, q)-fuzzy semi-open set (see [7]) (briefly, (p, q)-fs-open), if $\lambda \leq C_{\tau,\tau^*}(I_{\tau,\tau^*}(\lambda, p, q), p, q)$. λ is an (p, q)-fuzzy semi closed set (briefly, (p, q)-fs-closed), if $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \lambda$.
3. λ is an (p, q)-fuzzy α-open set (see[16]) (briefly, (p, q)-fα-open), if $\lambda \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(I_{\tau,\tau^*}(\lambda, p, q), p, q), p, q)$. λ is an (p, q)-fuzzy α-closed set (briefly, (p, q)-fα-closed), if $C_{\tau,\tau^*}(I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \lambda$.
4. λ is an (p, q)-generalized fuzzy-closed set (see [7]) (briefly, (p, q)-gf-closed), if $C_{\tau,\tau^*}(\lambda, p, q) \leq \mu$ whenever $\lambda \leq \mu$, $\tau(\mu) \geq p$ and $\tau^*(\mu) \leq q$. λ is called (p, q)-generalized fuzzy open (briefly, (p, q)-gf-open) iff $1-\lambda$ is (p, q)-gf-closed set.

5. λ is an (p, q)-generalized fuzzy semi closed (see [7]) (briefly, (p, q)-gfsc) if $C_{\tau,\tau^*}(\lambda, p, q) \leq \mu$, whenever $\lambda \leq \mu,\ \tau(\mu) \geq p\ and\ \tau^*(\mu) \leq q$. λ is called (p, q)- fuzzy generalized semi open (briefly, (p, q)-fgs-open) iff $1 - \lambda$ is (p, q)-fgsc set.

6. λ is an (p, q)-α-generalized fuzzy $closed$ (see [17] (briefly, (p, q)-αgf-closed) if $\alpha\, C_{\tau,\tau^*}(\lambda, p, q) \leq \mu$, whenever $\lambda \leq \mu$ and μ is (p, q)-$f\alpha$-open set.

(p, q)-fuzzy α^m-closed set

Now, we introduce the concept of (p, q)-fuzzy α^m-closed set in dfts.

Definition 3.1. Let $(X, \tau, \tau*)$ be a dfts, for each $\lambda, \mu \in I^X$, $p \epsilon I_{p0}$ and $q \in I_{q1}$, a fuzzy set λ is called (p, q)-fuzzy α^m-closed set (briefly, (p, q)-f α^m-closed) if $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \mu$, whenever $\lambda \leq \mu$ and μ is (p, q)-fα-open. λ is called (p, q)-fuzzy α^m-open iff 1-λ is (p, q)-fuzzy α^m- closed. From the definition of (p, q)-fα-closed we had reached directly that every (p,q)- fuzzy open sets is (p, q)-fα-closed and then get the next theorem.

Theorem 3.2.

Every (p, q)-fα-closed set is an (p, q)-fα^m-closed set**,** whenever $p \epsilon I_{p0}$ and $q \in I_{q1}$.

Proof: Assume that λ is an (p, q)-fα-closed set in I^X, $p \epsilon I_{p0}$ and $q \in I_{q1}$, then

$C_{\tau,\tau^*}(I_{\tau,\tau^*}(C_{\tau,\tau^*}((\lambda, p, q), p, q), p, q) \leq \lambda.$

Let $\tau(\mu) \geq p\ \ and\ \tau^*(\mu) \leq q$, such that $\lambda \leq \mu$. Since every (p, q)-fuzzy open set is an (p, q)-fα-open, then $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq I_{\tau,\tau^*}(\lambda, p, q) \leq \lambda \leq \mu$.

Therefore, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q),\ p, q) \leq \mu$, that is λ is an (p, q)-fα^m-closed set**.**

Remark 3.3. The converse of Theorem 3.2. is not true in general and can be show that by the following example.

Example 3.4. Let $X = \{m, n\}$, λ and μ are fuzzy sets defined as:

$$\lambda(m) = 0.4,\ \lambda(n) = 0.4,$$

$$\mu(m) = 0.6,\ \mu(n) = 0.7,$$

And, $$\gamma(m) = 0.6,\ \gamma(n) = 0.3.$$

Defined the dft $(\tau, \tau*)$ on X as follows , whenever $xt \in X$:

$$\tau(\lambda) = \begin{cases} 1, & if\ \lambda \in \{0,1\}, \\ \frac{1}{2}, & if\ \lambda(xt) = \lambda, \\ \frac{1}{3}, & if\ \lambda(xt) = \mu, \\ \frac{1}{4}, & if\ \lambda(xt) = \gamma, \\ 0, & otherwise \end{cases} \quad , \quad \tau^*(\lambda) = \begin{cases} 0, & if\ \lambda \in \{0,1\}, \\ \frac{1}{2}, & if\ \lambda(xt) = \lambda, \\ \frac{1}{3}, & if\ \lambda(xt) = \mu, \\ \frac{1}{4}, & if\ \lambda(xt) = \gamma, \\ 1, & otherwise \end{cases}$$

Then, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) = \lambda \leq \gamma = U$.

$\therefore$ λ is $(\frac{1}{2}, \frac{1}{2})$- f α^m-closed set, but $C_{\tau,\tau^*}(I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, \frac{1}{2}, \frac{1}{2}), \frac{1}{2}, \frac{1}{2}), \frac{1}{2}, \frac{1}{2}) = 1 - \mu \nleq \lambda$ which is not $(\frac{1}{2}, \frac{1}{2})$-fα-closed.

Theorem 3.5. The concept of (p, q)-fuzzy semi closed set and (p, q)-f α^m-closed set are equivalent.

Proof: Let λ be an (p, q)-fuzzy semi closed then, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \lambda$ and $\tau(\mu) \geq p$, $\tau^*(\mu) \leq q$ such that $\lambda \leq \mu$, $p \in I_{p0}$ and $q \in I_{q1}$. Since μ is (p, q)-fuzzy open set, then μ is an (p, q)-fα-open set $\Rightarrow C_{\tau,\tau^*}(I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \mu$, therefore λ is an (p, q)-fuzzy α^m-closed. The conversely is proof directly.

Remark 3.6. The concept of (p, q)-fp-closed set and (p, q)-f α^m-closed set are independent the following examples shows this states.

Example 3.7. Define (τ, τ∗) on X where, X = {m, n} as:

$$\tau(\lambda) = \begin{cases} 1, & if\ \lambda \in \{0,1\}, \\ \frac{1}{2}, & if\ \lambda(xt) = \lambda, \\ \frac{1}{3}, & if\ \lambda(xt) = \mu, \\ \frac{1}{4}, & if\ \lambda(xt) = \gamma, \\ 0, & otherwise \end{cases} \quad , \quad \tau^*(\lambda) = \begin{cases} 0, & if\ \lambda \in \{0,1\}, \\ \frac{1}{2}, & if\ \lambda(xt) = \lambda, \\ \frac{2}{3}, & if\ \lambda(xt) = \mu, \\ \frac{3}{4}, & if\ \lambda(xt) = \gamma, \\ 1, & otherwise \end{cases}$$

1- Take λ, μ and γ are fuzzy sets defined as:

$$\lambda(m) = 0.4, \qquad \lambda(n) = 0.4,$$
$$\mu(m) = 0.6, \qquad \mu(n) = 0.7,$$

$$\gamma(m) = 0.6, \qquad \gamma(n) = 0.3.$$

Then, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,p,q),\ p,q) \leq \cup$ where $\lambda \leq \gamma$ and $\cup = \gamma$.

Since $I_{\tau,\tau^*}(\mu^c, \frac{1}{2}, \frac{1}{2}) = \lambda \leq \gamma = \cup$.

That is, γ is an$(\frac{1}{2}, \frac{1}{2})$-f$\alpha^m$-closed.

But, $C_{\tau,\tau^*}\left(I_{\tau,\tau^*}\left(\lambda, \frac{1}{2}, \frac{1}{2}\right), \frac{1}{2}, \frac{1}{2}\right) = \mu^c \nleq \lambda$. So, λ is not $(\frac{1}{2}, \frac{1}{2})$-fp-closed set.

$\Rightarrow \lambda$ is $(\frac{1}{2}, \frac{1}{2})$-f α^m– closed, but not $(\frac{1}{2}, \frac{1}{2})$-fp-closed set.

2- Take λ, μ and γ to be fuzzy sets defined as follows:

$$\lambda(m) = 0.3, \qquad \lambda(n) = 0.4,$$
$$\mu(m) = 0.7, \qquad \mu(n) = 0.9,$$
$$\gamma(m) = 0.6, \qquad \gamma(n) = 0.4,$$
$$\beta(m) = 0.6, \qquad \beta(n) = 0.8.$$

So, $C_{\tau,\tau^*}\left(I_{\tau,\tau^*}(\beta,p,q),p,q\right) = C_{\tau,\tau^*}(I_{\tau,\tau^*}(0.6\ ,0.8\ ), \frac{1}{2}, \frac{1}{2}) \frac{1}{2}, \frac{1}{2}) = C_{\tau,\tau^*}(0, \frac{1}{2}, \frac{1}{2}) =$

$0 \leq (0.6,\ 0.8) < \beta$.

$\therefore \beta$ is $(\frac{1}{2}, \frac{1}{2})$ –fp-closed set.

If we take δ = (0.6, 0.3) $\Rightarrow \boldsymbol{\delta} \leq I_{\tau,\tau^*}\left(C_{\tau,\tau^*}\left(I_{\tau,\tau^*}(\delta,p,q),p,q\right),p,q\right)$

$$= I_{\tau,\tau^*}\left(C_{\tau,\tau^*}(0.6\ ,0.4), \frac{1}{2}, \frac{1}{2}\right), \frac{1}{2}, \frac{1}{2})$$

$$= I_{\tau,\tau^*}(1, \frac{1}{2}, \frac{1}{2}) = 1 \Rightarrow \delta \leq 1$$

$$\Rightarrow \boldsymbol{\delta} \ \boldsymbol{is} \left(\frac{1}{2}, \frac{1}{2}\right) - \boldsymbol{f\alpha}\text{-open which is equal}\cup$$

But, $\beta \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(0.6\ ,0.8\ ), \frac{1}{2}, \frac{1}{2}) \leq I_{\tau,\tau^*}(\lambda^c, \frac{1}{2}, \frac{1}{2}) = (0.7\ ,0.4\ ) \nleq \delta$

So, $\beta = (0.6\ ,0.8)$ is not $(\frac{1}{2}, \frac{1}{2})$-f α^m–closed set.

Theorem 3.8. Any fuzzy set λ is an (p, q)- $f\alpha^m$-closed set iff $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,p\ ,q),p,q) - \lambda$ contains no non- zero (p, q)-fα^m-closed set.

Proof: $\Rightarrow$ Let v be a non-zero an (p, q)-fα^m-closed subset of $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,p\ ,q),p,q)$, such that $v \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),p,q) - \lambda$, then $v \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),\ p,q) \wedge (1-\lambda)$.

Therefore, $v \le I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,\ q),p,q)$ and $v \le 1\text{-}\ \lambda$. Since 1-v is an (p, q)-f α^m-open set and λ is an (p, q)-fα^m- closed set, and since $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),\ p,q) \le 1 - v$.

Then, $v \le (\ I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),p,q))\ \wedge\ 1 - (\ I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,\ q),\ p,\ q)\ ) = 0$. So, $v = 0$ *this implies that* $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),\ p,q) - \lambda$ contain no non-zero (p, q)-fα^m- closed set.

$\Leftarrow$ Let $\lambda \le\ \mu$ be an (p, q)-fα^m-open set and assume that $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,\ \ q),\ p,q)$ is not contained in μ.

Then, 1-($I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p\ ,q),p,q)$) is a nonzero (p, q)-f$\alpha^m$-closed set and contained in $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),p,q) -\ \lambda$ which is a contradiction. Therefore $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,p\ ,q),p,q) \le\ \mu$ and hence λ is an (p, q)-f α^m- closed set.

Theorem 3.9. Let $\lambda \le 1_Y \le 1_X$, if λ is an (p, q)-fα^m-closed relative to 1_Y and 1_Y is an (p,q)-f-open set, then λ is an (p, q)-f α^m-closed set in (X, τ , τ^*) *whenever*, p $\epsilon \mathrm{I}_{p0}$, q$\in \mathrm{I}_{q1}$.

Proof: Let μ be an (p, q)-fα-open set in (X, τ , τ^*), p$\epsilon\mathrm{I}_{\mathrm{p}0}$ *and* q $\in \mathrm{I}_{\mathrm{q}1}$such that $\lambda \le \mu$.

Given that $\lambda \le 1_Y \le 1_X$, therefore $\lambda \le 1_Y$ and $\lambda \le \mu$. This implies $\lambda \le 1_Y \wedge \mu$.

Since λ is an (p, q)-f α^m-closed relative to 1_Y, then $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),\ p,q) \le\ \mu$.

But $1_Y \wedge\ I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),\ p,q) \le 1_Y \wedge \mu\ \Rightarrow\ 1_Y \wedge\ I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),p,q) \le\ \mu$.

Thus, $[\ 1_Y \wedge\ I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(\lambda,\ p,q),p,q\big)] \vee 1 - \big[I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(\lambda,p,q),p,q\big)\big]$

$$\le \mu \vee\ 1 - \ \big[I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,p,q),\ p,q)\big].$$

$\Rightarrow [\ 1_{\mathrm{Y}} \vee 1 - [\mathrm{I}_{\tau,\tau^*}(\mathrm{C}_{\tau,\tau^*}(\lambda,\mathrm{p}\ ,\mathrm{q}),\mathrm{p},\mathrm{q})] \wedge \mathrm{I}_{\tau,\tau^*}(\mathrm{C}_{\tau,\tau^*}(\lambda,\ \mathrm{p},\mathrm{q}),\mathrm{p},\mathrm{q})\ ] \vee 1 - [\mathrm{I}_{\tau,\tau^*}(\mathrm{C}_{\tau,\tau^*}(\lambda,$ $\mathrm{p},\mathrm{q}),\ \mathrm{p},\mathrm{q})] \ \le\ \mu\ \vee\ 1 - \ [\mathrm{I}_{\tau,\tau^*}(\mathrm{C}_{\tau,\tau^*}(\lambda,\mathrm{p},\mathrm{q}),\mathrm{p},\mathrm{q})]$.

Therefore, $(1_Y < 1 - \big[I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(\lambda,p,q),\ p,q\big)\big])\ \ \le\ \mu\ \vee\ 1 - \ \big[I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(\lambda,p,q),p,q\big)\big]$.

Since 1_Y is an (p, q)-f α^m-closed set in 1_X.

And,

$$I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),p,q) \le\ \mu \vee\ 1 - \ \big[I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(\lambda,\ p,\ q),\ p,q\big)\big].$$

Also, $\lambda \le 1_Y \Rightarrow I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),p,q) \le\ I_{\tau,\tau^*}(C_{\tau,\tau^*}(1_Y,\ p,q),p,q)$.

Thus, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,\ p,q),p,q) \le\ I_{\tau,\tau^*}(C_{\tau,\tau^*}(1_Y,\ p,q),p,q) \le\ \mu \vee 1 - \ \big[I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(\lambda,$ $p,q),p,q\big)\big]$.

Therefore, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \mu$.

Since $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q)$ is not contained in $1 - [I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q)]$

So, λ an (p, q)-fα^m-closed relative to 1_X.

Theorem 3.10. If λ is an (p, q)-fα^m-closed set and $\lambda \leq v \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q)$, then v is an (p, q)-fα^m-closed set whenever $p \epsilon I_{p0}$ *and* $q \in I_{q1}$.

Proof: Let λ be an (p, q)-fα^m-closed set, $p \epsilon I_0$ and $q \in I_1$ such that $\lambda \leq v \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q)$, also let μ be an (p, q)-fα^m-*open set* of such that $v \leq \mu$. Since λ is an (p, q)-fα^m-closed set, then we have $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \mu$, where $\lambda \leq \mu$. Since $\lambda \leq v$ and $v \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q)$ then,

$$\begin{aligned} I_{\tau,\tau^*}(C_{\tau,\tau^*}(v, p, q), p, q) &\leq (I_{\tau,\tau^*}(C_{\tau,\tau^*}(I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q), p, q), p, q) \\ &\leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \\ &\leq \mu \end{aligned}$$

Therefore, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(v, p, q), p, q) \leq \mu$. So, v is an (p, q)-fα^m-closed set in X.

Theorem 3.11. Let λ and v are two (p, q)-fuzzy sets in (X, τ, τ^*) such that $p \epsilon I_0$, $q \in I_1$, then,

1- If λ is an (p, q)-fα^m-closed set and $\tau(1 - v) \geq p$ *and* $\tau^*(1 - v) \leq q$, then $\lambda \wedge v$ is an (p, q)-fα^m-closed *set*.

2- If λ and v are two (p, q)-fα^m-closed sets, then $\lambda \wedge v$ is (p, q)-fα^m-closed *set*.

Proof: 1- let λ be an (p, q)-fα^m-closed set and v be any $\tau(1 - v) \geq p$ *and* $\tau^*(1 - v) \leq q$ where $p \epsilon I_{p0}$, $q \in I_{q1}$. Since λ is an (p, q)-fα^m-closed set so, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \mu$ where $\lambda \leq \mu$ and μ is an (p, q)-fα^m-open set.

To show, $\lambda \wedge v$ is an (p, q)-fα^m-closed set, it is enough to proof that $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda \wedge v, p, q), p, q) \leq \mu$ where $\lambda \wedge v \leq \mu$ *and* μ is an (p, q)-fα-open set

Let $\rho = 1 - v$, then $\lambda \leq \mu \vee \rho$. Since $\tau(\rho) \geq p$ *and* $\tau^*(\rho) \leq q$, $\mu \vee \rho$ is an (p, q)-fα^m-open set and λ is an (p, q)-fα^m-closed set so, $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \leq \mu \vee \rho$.

Now,

$$\begin{aligned} I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda \wedge v, p, q), p, q) &\leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \wedge I_{\tau,\tau^*}(C_{\tau,\tau^*}(v, p, q), p, q) \\ &\leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \wedge v \end{aligned}$$

$\leq (\mu \vee \rho) \wedge \upsilon$

$\leq (\mu \wedge \upsilon) \vee (\rho \wedge \upsilon)$

$\leq (\mu \wedge \upsilon) \vee 0 \leq \mu.$

$\Rightarrow \lambda \wedge \upsilon$ is an (p, q)-fα^m-closed set.

2-Let λ and υ are two (p, q)-fα^m-closed set whenever pϵI_0 *and* q $\in I_1$.

Consider μ be an (p, q)-fα^m-open set in X such that $\lambda \wedge \upsilon \leq \mu$.

Now,

$I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda \wedge \upsilon, p, q), p, q) \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, p, q), p, q) \wedge I_{\tau,\tau^*}(C_{\tau,\tau^*}(\upsilon, p, q), p, q) \leq \mu.$

Hence $\lambda \wedge \upsilon$ is an (p, q)-fα^m-closed set.

Remark 3.12. The union of two (p, q)-fα^m-closed sets need not to be (p, q)-fα^m-closed set. The next example showing this case.

Example 3.13. Take **Example 3.4**. and

$\lambda(m) = 0.4, \quad \lambda(n) = 0.4,$

$\mu(m) = 0.6, \quad \mu(n) = 0.7,$

$\gamma(m) = 0.6, \quad \gamma(n) = 0.3.$

So, $I_{\tau,\tau^*}\left(C_{\tau,\tau^*}\left((0.4, 0.4), \frac{1}{2}, \frac{1}{2}\right), \frac{1}{2}, \frac{1}{2}\right) \leq I_{\tau,\tau^*}(\mu^c, \frac{1}{2}, \frac{1}{2}) = \lambda \leq \gamma.$

That is λ *is* $(\frac{1}{2}, \frac{1}{2})$-f$\alpha^m$-closed set.

Also, $I_{\tau,\tau^*}\left(C_{\tau,\tau^*}\left(\mu, \frac{1}{2}, \frac{1}{2}\right), \frac{1}{2}, \frac{1}{2}\right) \leq I_{\tau,\tau^*}(C_{\tau,\tau^*}((0.6, 0.7), \frac{1}{2}, \frac{1}{2}), \frac{1}{2}, \frac{1}{2}) = I_{\tau,\tau^*}(\lambda^c, \frac{1}{2}, \frac{1}{2}) = \mu \leq \gamma$

So, μ *is* $\left(\frac{1}{2}, \frac{1}{2}\right)$-f$\alpha^m$-closed set.

We know that, $\lambda \vee \mu = (0.4, 0.4) \vee (0.6, 0.7) = (0.6, 0.4).$

And $I_{\tau,\tau^*}\left(C_{\tau,\tau^*}(0.6, 0.4), \frac{1}{2}, \frac{1}{2}\right), \frac{1}{2}, \frac{1}{2}) \leq I_{\tau,\tau^*}(1, \frac{1}{2}, \frac{1}{2}) = 1 \nleq \gamma$ which is refers to U in the definition, this implies that $\lambda \vee \mu$ is not $\left(\frac{1}{2}, \frac{1}{2}\right)$- f$\alpha^m$-closed set.

$\Rightarrow \lambda$ *and* μ are $\left(\frac{1}{2}, \frac{1}{2}\right)$-f$\alpha^m$-closed set, but $\lambda \vee \mu$ is not $\left(\frac{1}{2}, \frac{1}{2}\right)$-f$\alpha^m$-closed set.

Remark 3.14. Every (p, q)-fuzzy closed set is (p, q)-fα^m-closed set whenever p ϵI_{p0} *and* q $\in I_{q1}$. But the contrariwise is not true generally.

Example 3.15. Take Example 3.13. then we get:

$I_{\tau,\tau^*}(C_{\tau,\tau^*}((\lambda),\frac{1}{2},\frac{1}{2}),\frac{1}{2},\frac{1}{2}) \leq I_{\tau,\tau^*}(\mu^c,\frac{1}{2},\frac{1}{2}) = \lambda \leq \gamma = \cup$

This is λ is $(\frac{1}{2},\frac{1}{2})$-f$\alpha^m$-closed set, but not $(\frac{1}{2},\frac{1}{2})$-fc set.

Definition 3.16. If (X, τ , τ^*) be a dfts. So, for each λ ,$\mu \in I^x$, pϵI_{p0} *and* q $\in I_{q1}$, we have the α^m – Closure of λ is defined as:

$$\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) = \wedge\{\mu \in I^x : \lambda \leq \mu, \mu \text{ is ((p, q)-f}\alpha^m\text{-closed}\}$$

Theorem 3.17. If (X , τ,τ^*) be a dfts, then for each p ϵI_{p0}, q $\in I_{q1}$ and λ, $\mu \in I^x$, the operator $\alpha^m\, C_{\tau,\tau^*}$: $I^x \times I_{p0} \times I_{q1} \to I^x$ satisfy the following statements:

(C1) $\alpha^m\, C_{\tau,\tau^*}(0,p,q) = 0$, $\alpha^m\, C_{\tau,\tau^*}(1,p,q) = 1$.

(C2) $\lambda \leq \alpha^m\, C_{\tau,\tau^*}(\lambda,p,q)$,

(C3) If $\lambda \leq \mu$, then $\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) \leq \alpha^m\, C_{\tau,\tau^*}(\mu,p,q)$,

(C4) λ *is* (p, q)-fα^m-closed iff $\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) = \lambda$

(C5) If v *is* (p, q)- fα^m-open, then $vq\,\lambda$ iff $vq\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q)$,

(C6) $\alpha^m\, C_{\tau,\tau^*}(\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q)$, p, q) = $\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q)$,

(C7) $\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) \vee \alpha^m\, C_{\tau,\tau^*}(\mu,p,q) \leq \alpha^m\, C_{\tau,\tau^*}(\lambda \vee \mu,p,q)$, If the intersection of two (p, q)-fα^m-open is an *is* (p, q)-fα^m-open, then
$\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) \vee \alpha^m\, C_{\tau,\tau^*}(\mu,p,q) = \alpha^m\, C_{\tau,\tau^*}(\lambda \vee \mu,p,q)$.

Proof: Let λ , $\mu \in I^x$, pϵI_{p0} *and* q $\in I_{q1}$.

(C1) $\alpha^m\, C_{\tau,\tau^*}(0,p,q) = \wedge\{\mu \in I^x : 0 \leq \mu, \mu$ is (p, q)-fα^m-closed} = 0, and
$\alpha^m\, C_{\tau,\tau^*}(1,p,q) = \wedge\{\mu \in I^x : 1 \leq \mu, \mu$ is (p, q)-fα^m-closed} = 1,

(C2) $\lambda \leq \wedge\{\mu \in I^x : \lambda \leq \mu, \mu$ is (p, q)-fα^m-closed} = $\alpha^m\, C_{\tau,\tau^*}(\lambda, p,q)$,

(C3) we know, $\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) = \wedge\{\mu \in I^x : \lambda \leq \mu, \mu$ is (p, q)-fα^m-closed}

Since $\lambda \leq \mu$ then, $\wedge\{\mu \in I^x : \lambda \leq \mu, \mu$ is (p, q)-fα^m-closed} $\leq \wedge\{v \in I^x : \mu \leq v, v$ is (p, q)-fα^m-closed} $\Rightarrow \alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) \leq \alpha^m\, C_{\tau,\tau^*}(\mu,p,q)$.

(C4) if λ is (p, q)-fα^m-closed, then $\alpha^m\, C_{\tau,\tau^*}(\lambda,p,q) = \wedge\{\mu \in I^x : \lambda \leq \mu, \mu$ is (p, q)-fα^m-closed} and $\lambda \leq \alpha^m\, C_{\tau,\tau^*}(\lambda,p,q)$, but λ is necessarily the smallest set.

Thus $\lambda = \wedge\{\mu \in I^x : \lambda \leq \mu, \mu$ is (p, q)-fα^m-closed}, therefore $\lambda = \alpha^m\, C_{\tau,\tau^*}(\lambda,p,q)$.

$\Leftarrow$, Let μ be any subset in X then,

$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) = \wedge\{\mu \in I^x: \lambda \le \mu, \mu$ is (p, q)-fα^m-closed$\}$, but $\lambda \le \wedge\{\mu \in I^x: \lambda \le \mu, \mu$ is (p, q)-fα^m-closed$\}$. Since the intersection of all (p, q)-fc set is (p, q)-fc, it follows that $\alpha^m C_{\tau,\tau^*}(\lambda, p, q)$ is (p, q)-fc in X. Suppose that $\lambda = \alpha^m C_{\tau,\tau^*}(\lambda, p, q)$, and we have $\alpha^m C_{\tau,\tau^*}(\lambda, p, q)$ is (p, q)-fc $\Rightarrow \lambda$ is (p, q)-fα^m-closed set.

(C5) Let $v\bar{q}\lambda$ where v be (p, q)-fα^m-open set. Then $\lambda \le 1 - v$, if we take $\alpha^m C_{\tau,\tau^*}$ so we get, $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \le \alpha^m C_{\tau,\tau^*}(1 - v, p, q)$.

But,

$$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \le \alpha^m C_{\tau,\tau^*}(1 - v, p, q) = 1 - v$$

by (C4) we get $v\bar{q}\alpha^m C_{\tau,\tau^*}(\lambda, p, q)$ which is contradiction,

then $vq\lambda$ iff $vq\, \alpha^m C_{\tau,\tau^*}(\lambda, p, q)$.

(C6) Since $\lambda = \alpha^m C_{\tau,\tau^*}(\lambda, p, q)$ we take $\alpha^m C_{\tau,\tau^*}$ two side we get,

$$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) = \alpha^m C_{\tau,\tau^*}(\alpha^m C_{\tau,\tau^*}(\lambda, p, q), p, q)$$

(C7) Let λ and μ are (p, q)-fα^m-closed set, then

$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) = \wedge\{v \in I^x: \lambda \le v, v$ is (p, q)-fα^m-closed$\}$……..(1)

$\alpha^m C_{\tau,\tau^*}(\mu, p, q) = \wedge\{v \in I^x: \mu \le v, v$ is (p, q)-fα^m-closed$\}$……. (2)

Then by (1) and (2) we get, $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \vee \alpha^m C_{\tau,\tau^*}(\mu, p, q) =$

$[\wedge\{v \in I^x: \lambda \le v, v$ is (p, q)-fα^m-closed$\}] \vee [\wedge\{v \in I^x: \mu \le v, v$ is (p, q)-fα^m-closed$\}]$

$\le \alpha^m C_{\tau,\tau^*}(\lambda \vee \mu, p, q)$ ……(3)

But if we get λ and μ are (p, q)-fα^m-open *set* and $\lambda \wedge \mu$ are (p, q)-fα^m-open set, then by (1) and (2) we get,

$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \vee \alpha^m C_{\tau,\tau^*}(\mu, p, q) = \alpha^m C_{\tau,\tau^*}(\lambda \vee \mu, p, q)$.

Theorem 3.18. If (X, τ, τ^*) be a dfts, then for each $p \in I_0$, $q \in I_1$ and $\mu, \lambda \in I^x$, we define an operator $\alpha^m I_{\tau,\tau^*}: I^x \times I_{p0} \times I_{q1} \to I^x$ as:

$\alpha^m I_{\tau,\tau^*}(\lambda, p, q) = \vee\{\mu \in I^x: \mu \le \lambda, \mu$ *is* (p, q)-fα^m-open *set*$\}$.

The operator I_{τ,τ^*} satisfies the following statements:

(I1) $\alpha^m I_{\tau,\tau^*}(0, p, q) = 0$, $\alpha^m I_{\tau,\tau^*}(1, p, q) = 1$.

(I2) $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \le \lambda$.

(I3) If λ is an (p, q)-fα^m-open set, then $\lambda = \alpha^m I_{\tau,\tau^*}(\lambda, p, q)$ and

$\alpha^m I_{\tau,\tau^*}(\alpha^m I_{\tau,\tau^*}(\lambda, p, q), p, q) = \alpha^m I_{\tau,\tau^*}(\lambda, p, q)$

(I4) If $\lambda \le \mu$, then $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \le \alpha^m I_{\tau,\tau^*}(\mu, p, q)$,

(I5) $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \wedge \alpha^m I_{\tau,\tau^*}(\mu, p, q) \ge \alpha^m I_{\tau,\tau^*}(\lambda \wedge \mu, p, q)$. If the intersection of two (p, q)-fα^m-open sets are (p, q)-fα^m-open set, then

$$\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \wedge \alpha^m I_{\tau,\tau^*}(\mu, p, q) = \alpha^m I_{\tau,\tau^*}(\lambda \wedge \mu, p, q).$$

Proof: $p \epsilon I_{p0}$ and $q \in I_{q1}$ and $\mu, \lambda \in I^x$

(I1). $\alpha^m I_{\tau,\tau^*}(0, p, q) = \vee\{\mu \in I^x : \mu \le 0, \mu\ is$ (p, q)-fα^m-open $set\}$ = 0,

And $\alpha^m I_{\tau,\tau^*}(1, p, q) = \vee\{\mu \in I^x : \mu \le 1, \mu\ is$ (p, q)-fα^m-open $set\}$ = 1.

(I2). $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) = \vee\{\mu \in I^x : \mu \le \lambda, \mu\ is$ (p, q)-fα^m-open $set\} \le \lambda$, then $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \le \lambda$.

(I3). We must prove that $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \le \lambda$ & $\lambda \le \alpha^m I_{\tau,\tau^*}(\lambda, p, q)$.

By (I2), we get $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \le \lambda$ .... (1)

Now to prove $\lambda \le \alpha^m I_{\tau,\tau^*}(\lambda, p, q)$, for all $\lambda \le \lambda$, $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \le \lambda$.

So, we get, $\lambda \le \alpha^m I_{\tau,\tau^*}(\lambda, p, q)$ .... (2)

From (1) & (2) we get, $\lambda = \alpha^m I_{\tau,\tau^*}(\lambda, p, q)$.

(I4). We have, $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) = \vee\{\mu \in I^x : \mu \le \lambda, \mu\ is$ (p, q)-fα^m-open $set\}$.

And since $\lambda \le \mu$ then we get,

$\vee\{\mu \in I^x : \mu \le \lambda, \mu\ is$ (p, q)-fα^m-open $set\} \le \vee\{v \in I^x : v \le \mu, v\ is$ (p, q)-fα^m-open $set\}$

$\Rightarrow \alpha^m I_{\tau,\tau^*}(\lambda, p, q) \le \alpha^m I_{\tau,\tau^*}(\mu, p, q)$.

(I5) Let λ and μ are two (r, s)-fα^m-open set,

Then, $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) = \vee\{v \in I^x : v \le \lambda, v\ is$ (p, q)-fα^m-open $set\}$..........(1)

Also, $\alpha^m I_{\tau,\tau^*}(\mu, p, q) = \vee\{v \in I^x : v \le \mu, v\ is$ (p, q)-fα^m-open $set\}$.........(2)

Then by (1) and (2) We get, $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \wedge \alpha^m I_{\tau,\tau^*}(\mu, p, q) =$

$[\vee\{v \in I^x : v \le \lambda, v\ is$ (p, q)-fα^m-open $set\}] \wedge [\vee\{v \in I^x : v \le \mu, v\ is$ (p, q)-fα^m-open $set\}] \ge \vee\{v \in I^x : v \le \lambda \wedge \mu, v\ is$ (p, q)-fα^m-open $set\}$

$\ge \alpha^m I_{\tau,\tau^*}((\lambda \wedge \mu, p, q)$.

But, if μ and λ are (p, q)-fα^m-open set and $\lambda \wedge \mu$ is (p, q)-fα^m-open set, then by (1) and (2) we get,

$$\alpha^m I_{\tau,\tau^*}(\lambda,p,q) \wedge \alpha^m I_{\tau,\tau^*}(\mu,p,q) = \alpha^m I_{\tau,\tau^*}(\lambda \wedge \mu, p,q).$$

Theorem 3.19. Let (X , τ,τ^*) be a dfts. For all fuzzy set $\lambda \in I^x$, pϵI_{p0} and q $\in I_{q1}$, the following statements are correct:

(1) $\alpha^m I_{\tau,\tau^*}(1-\lambda,p,q) = 1-\alpha^m C_{\tau,\tau^*}(\lambda,p,q)$.

(2) $\alpha^m C_{\tau,\tau^*}$ (1- λ,p,q) = 1 − $\alpha^m I_{\tau,\tau^*}(\lambda,p,q)$.

(3) If $\alpha^m I_{\tau,\tau^*}(\alpha^m C_{\tau,\tau^*}(\lambda,p,q)$, p, q) = λ , then, $\alpha^m C_{\tau,\tau^*}$ ($\alpha^m I_{\tau,\tau^*}(1-\lambda,p,q)$,p, q) =1 − λ .

Proof: (**1**) $\alpha^m I_{\tau,\tau^*}(1-\lambda,p,q) = \vee\{\mu \in I^x: \mu \le 1-\lambda,\ \mu\ is$ (p, q)-fα^m-open $set\}$

= 1- $\wedge$ { $\mu \in I^x: \lambda \le \mu,\ \mu\ is$ (p, q)-fα^m-closed $set\}$

= $1-\alpha^m C_{\tau,\tau^*}(\lambda,p,q)$.

(2) $\alpha^m C_{\tau,\tau^*}$ (1- λ,p,q) = $\wedge$ { $\mu \in I^x: 1-\lambda \le \mu,\ \mu\ is$ (p, q)-fα^m-closed $set\}$

= 1- $\vee\{\mu \in I^x: \mu \le \lambda,\ \mu\ is$ (p, q)-fα^m-open $set\}$

= 1 − $\alpha^m I_{\tau,\tau^*}(\lambda,p,q)$.

(3) $\alpha^m C_{\tau,\tau^*}$ ($\alpha^m I_{\tau,\tau^*}(1-\lambda,p,q)$, p, q) = $\wedge$ {$\mu \in I^x$ ({ $\vee\{\mu \in I^x: \mu \le 1-\lambda,\ \mu\ is$ (p, q)-fα^m-open $set\}$}) : 1- $\lambda \le \mu$: μ is (p, q)-fα^m-closed $set\}$ =
$\wedge$ {$\mu \in I^x$({$\vee\{\mu \in I^x: \mu \le 1-\mu, \mu\ is$ (p, q)-fα^m-open set}): 1- $\mu \le \lambda$, μ is (p, q)-fα^m-closed}= 1-$\vee\{\mu \in I^x \wedge \{\mu \in I^x: \lambda \le \mu,\ \mu\ is$ (p, q)-fα^m-closed set}): $\mu \le \lambda$, μ is (p, q)-fα^m-open set}.

Since $\alpha^m I_{\tau,\tau^*}(\alpha^m C_{\tau,\tau^*}(\lambda,p,q)$, p, q) = λ, so

1-$\vee\{\mu \in I^x \wedge \{\mu \in I^x: \lambda \le \mu,\ \mu\ is$ (p, q)-fα^m-closed set}): $\mu \le \lambda$, μ is (p, q)-fα^m-open set}
=1 − λ .

$\therefore \alpha^m C_{\tau,\tau^*}$ ($\alpha^m I_{\tau,\tau^*}(1-\lambda,p,q)$,p , q) =1 − λ.

(p, q)- α^m- generalized fuzzy-closed

Now, we introduce new class of closed sets, this concept is said to be (p, q)-α^m-generalized fuzzy-closed set in double fuzzy topological spaces.

Definition 4.1. If (X , τ,τ^*) be a dfts then, for each λ, $\mu \in I^x$ and p ϵI_{p0}, q $\in I_{q1}$, the fuzzy set λ is said to be (p, q)- α^m- generalized fuzzy-closed (briefly, (p, q)-α^m- gf-closed). If $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \leq \mu$ such that $\lambda \leq \mu$ and μ is (p, q)-fα^m-open set. λ is called (p, q)-α^m-generalized fuzzy-open (briefly, (p, q)-α^m- gf-open) iff $1-\lambda$ is an (p, q)-α^m-gfc set.

Remark 4.2. Every (p, q)-fuzzyα^m-closed set is an (p, q)-α^mgeneralized fuzzy-closed set**,** but the converse is not correct in general and we can showing by the following example.

Example 4.3. Take Example 3.7 (2) so we get, $C_{\tau,\tau^*}(\beta, \frac{1}{2}, \frac{1}{2}) = 1 \leq 1$.

So, β is an $(\frac{1}{2}, \frac{1}{2})$- α^m- gf-closed set

Now, if we put δ = (0.6, 0.3) is $(\frac{1}{2}, \frac{1}{2})$-α open then, β ≤ δ ⇒ β is not $(\frac{1}{2}, \frac{1}{2})$- f$\alpha^m$-closed set.

Theorem 4.4. Let (X, τ, τ^*) be a dfts, $\lambda \in I^x$, $p \in Ip_0$ *and* $q \in I_{q1}$. λ is an (p, q)-α^m- gf-open *set* iff $\mu \leq I_{\tau,\tau^*}(\lambda, p, q)$ whenever, $\mu \leq \lambda$ *and* μ *is* an (p, q)--fα^m-closed set.

Proof: ⇒ Let λ be an (p, q)-α^m- gf-open set in X and let μ be any (p, q)-fα^m-closed set in X such that $\mu \leq \lambda$ *and* μ *is an* (p, q)-fα^m-open, so 1- λ is an (p, q)- α^m- gf- closed. Therefore, for all (p, q)-fuzzy open set υ say υ=1- $\mu \in I^x$, we get 1- $\lambda \leq 1-\mu$, then

$$C_{\tau,\tau^*}(1-\lambda, p, q) \leq 1-\mu.$$

So, $1-(1-\mu) = \mu \leq 1-(C_{\tau,\tau^*}(1-\lambda, p, q)) = I_{\tau,\tau^*}(\lambda, p, q)$.

⇐ let μ be an (p, q)-fα^m-closed set, so for each $\mu \in I^x$, such that $\mu \leq \lambda$, $\tau(\mu) \geq p$ and $\tau^*(\mu) \leq q$. Now, $\mu \leq I_{\tau,\tau^*}(\lambda, p, q)$, if λ *is* an (p, q)-α^m- gf-open set ,this implies 1- λ is an (p, q)-α^m-gf-closed set, take $v \in I^x$ such that 1- $\lambda \leq v$, since $v \in I^x$, then 1- v is an (p, q)-fuzzy α^m-closed set and 1- $v \leq \lambda$, so by hypothesis 1- $v \leq I_{\tau,\tau^*}(\lambda, p, q)$.

Therefore, 1-$I_{\tau,\tau^*}(\lambda, p, q) = C_{\tau,\tau^*}(\lambda, p, q) \leq 1-(1-v) = v$, so by the definition of complement, we get that 1- λ is an (p, q)-α^m- gf-closed set.

Proposition 4.5. If λ is (p , q)-fα^m-closed set in a dfts (X, τ, τ^*), then λ is an (p, q)-α^m- gf closed set, for each $p \in I_{p0}$ and $q \in I_{q1}$.

Proof: It is obvious.

Theorem 4.6. If λ is (p, q)-α^m-gf closed set in a dfts (X, τ, τ^*) and $\lambda \leq \boldsymbol{\mu} \leq C_{\tau,\tau^*}$ (λ, p, q),

then $\boldsymbol{\mu}$ is an (p, q)-α^m-gf closed set, for each $p \in I_{p0}$ *and* $q \in I_{q1}$.

Proof: Assume that v is an (p, q)-fα^m-open set in X and $\boldsymbol{\mu} \leq v, then\ \lambda \leq v. but\ \lambda \in I^x$ is an (p, q)-α^m-gf closed set $\Rightarrow \alpha^m C_{\tau,\tau^*}(\lambda, p, q) \leq v$. From Theorem 4-4 we get,

$$\alpha^m C_{\tau,\tau^*}(\boldsymbol{\mu}, p, q) \leq \alpha^m C_{\tau,\tau^*}(\lambda, p, q),$$

and $\boldsymbol{\mu} \leq v$ then, $\alpha^m C_{\tau,\tau^*}(\boldsymbol{\mu}, p, q) \leq v$, so $\boldsymbol{\mu}$ is an (p, q)-α^m-gf closed set.

Theorem 4.7. If (X, τ, τ^*) be a dfts such that λ is an (p, q)-α^m-gf -open set and $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \leq \mu$ where $\mu \leq \lambda$, then μ is an (p, q)-α^m-gf -open set.

Proof. Assume that v is an (p, q)-fα^m-closed set in X and $v \leq \mu$ for each, $p \epsilon I_{p0}$ and $q \in I_{q1}$, so $v \leq \lambda$. But $\lambda \in I^x$ is an (p, q)-α^m-gf-open set $\Rightarrow v \leq \alpha^m I_{\tau,\tau^*}(\lambda, p, q)$, and hence $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \leq \alpha^m I_{\tau,\tau^*}(\mu, p, q)$, so $v \leq \alpha^m I_{\tau,\tau^*}(\mu, p, q)$, and hence μ is an (p, q)-α^m-gf -open set.

Definition 4.8. If (X, τ, τ^*) be a dfts, so a fuzzy set λ is said to be an (p, q)-fuzzyα^m-clopen (briefly , (p, q)-α^m-gf –clopen) set iff λ is an (p, q)-fα^m-open ((p, q)-α^m-gf –open) and ((p, q)-fα^m-closed ((p, q)-α^m-gf –closed) set whenever, $p \epsilon I_{p0}$ and $q \in I_{q1}$.

Theorem 4.9. Let (X, τ, τ^*) be a dfts and $\lambda \in I^x$, $p \epsilon I_{p0}\ and\ q \in I_{q1}$. If λ is an (p, q)-α^m-gf-clopen set, then $\lambda\ be$ an (p, q)-f α^m-closed set.

Proof: Suppose $\lambda\ and\ \mu \in I^x$ where, μ is an (p, q)-$f\alpha^m$-open such that $\lambda \leq \mu$, $p\ \epsilon I_{p0}$ and $q \in I_{q1}$. By the definition of complement, we get λ is an (p, q)-α^m-gf-closed, so $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) = \lambda \leq \mu$. But, λ is an (p, q)-α^m-gf- clopen set, That is $\lambda\ is\ an$ (p, q)-fα^m-closed set.

Theorem 4.10. Let (X, τ, τ^*) be a dfts whenever $\lambda \in I^x$, $p \epsilon I_{p0}\ and\ q \in I_{q1}$. If λ is both (p, q)-fα^m-open set and (p, q)-α^m-gf-closed set, then $\lambda\ is$ (p, q)-fα^m-closed set.

Proof: Suppose $\lambda\ is$ (p, q)-fα^m-open set and (p, q)-α^m-gf-closed set such that $\lambda \leq \lambda$, $p \epsilon I_{p0}$ and $q \in I_{q1}$ Then $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \leq \lambda$. But, $\lambda \leq \alpha^m C_{\tau,\tau^*}(\lambda, p, q)$, therefore $\lambda = \alpha^m C_{\tau,\tau^*}(\lambda, p, q)$. Hence, λ is (p, q)-fα^m-closed set.

Theorem 4.11. Let (X, τ, τ^*) be a dfts . Then each (p, q)-fα^m-open set is an (p, q)-fα^m-closed set iff every (p, q)-fuzzy subset in X is (p, q)-α^m-gf-closed set, $p \epsilon I_{p0}\ and\ q \in I_{q1}$.

Proof: Let μ be an (p, q)-fα^m-open set with $\lambda\ be$ (p, q)-fuzzy subset of X such that $\lambda \leq \mu$, but μ is an (p, q)-fα^m-closed set, $p \in I_{p0}$ *and* $q \in I_{q1}$. By Theorem 3.17 (C3) we get,

$$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \leq \alpha^m C_{\tau,\tau^*}(\mu, p, q).$$

Again by Theorem 3.17 (C3) we get, $\lambda \leq \alpha^m C_{\tau,\tau^*}(\lambda, p, q)$, then

$$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \leq \alpha^m C_{\tau,\tau^*}(\mu, p, q) \leq \mu$$

$$\Rightarrow \lambda \text{ is an (p, q)-}\alpha^m\text{-gf-closed set}.$$

$\Leftarrow$ let μ be any (p, q)-α^m-gf-closed set an (p, q)-fα^m-open set, but we have $\mu \leq \mu$.

Then by Theorem 3.17 (C2) we have, $\mu \leq \alpha^m C_{\tau,\tau^*}(\mu, p, q)$. So, $\alpha^m C_{\tau,\tau^*}(\mu, p, q) = \mu$.

Hence, μ is an (p, q)-fα^m-closed set.

Theorem 4.12. If $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \vee \alpha^m C_{\tau,\tau^*}(\mu, p, q) = \alpha^m C_{\tau,\tau^*}(\lambda \vee \mu, p, q)$. Then $\lambda \vee \mu$ is an (p, q)-α^m-gf-closed.

Proof: Let λ and μ are (p, q)-α^m-gf-closed *set and* v *be an* (p, q)-fα^m-open set such that $\lambda \vee \mu \leq v$, $p \in I_{p0}$ *and* $q \in I_{q1}$. Thus either, $\lambda \leq v$ *or* $\mu \leq v$, but

$$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \leq v \text{ and } \alpha^m C_{\tau,\tau^*}(\mu, p, q) \leq v.$$

Also by hypothesis,

$$\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \vee \alpha^m C_{\tau,\tau^*}(\mu, p, q) = \alpha^m C_{\tau,\tau^*}(\lambda \vee \mu, p, q)$$

$$\Rightarrow \alpha^m C_{\tau,\tau^*}(\lambda \vee \mu, p, q) \leq v.$$

Then $\lambda \vee \mu$ is an (p, q)-α^m-gf-closed set.

Theorem 4.13. If $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \wedge \alpha^m I_{\tau,\tau^*}(\mu, p, q) = \alpha^m I_{\tau,\tau^*}(\lambda \wedge \mu, p, q)$. Then, $\lambda \wedge \mu$ is an (p, q)-α^m-gf-open set.

Proof: Let λ *and* μ are (p, q)-α^m-gf-open set *and* v *be an* (p, q)-fα^m-closed set for each, $p \in I_{p0}$ *and* $q \in I_{q1}$, $v \leq \lambda$ *and* $v \leq \mu$. But

$$v \leq \lambda \wedge \mu,\ v \leq \alpha^m I_{\tau,\tau^*}(\lambda, p, q) \text{ and } V \leq \alpha^m I_{\tau,\tau^*}(\mu, p, q).$$

Also by hypothesis, $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \wedge \alpha^m I_{\tau,\tau^*}(\mu, p, q) = \alpha^m I_{\tau,\tau^*}(\lambda \wedge \mu, p, q)$

$$\Rightarrow v \leq \alpha^m I_{\tau,\tau^*}(\lambda \wedge \mu, p, q)$$

So, $\lambda \wedge \mu$ is an (p, q)-α^m-gf-open set.

Definition 4.14. If (X, τ, τ^*) be a dfts, then a fuzzy point z_t is called an (p, q)-fuzzy just-α^m-closed if $\alpha^m C_{\tau,\tau^*}(z_t, p, q)$ is a fuzzy point, $p \in I_{p0}$ *and* $q \in I_{q1}$.

Theorem 4-15: Let z_t and z_{t_0} are two fuzzy points in a dfts (X, τ, τ^*) such that $t < t_0$ and z_{t_0} is an (p, q)-fα^m-open. Then z_t is an (p, q)-fuzzy just –α^m-closed if z_t is an (p, q)-α^m-gf-closed set, whenever pϵI_{p0} and $q \in I_{q1}$.

Proof: let $z_t < z_{t_0}$ where z_{t_0} is an (p, q)-fα^m-open, and let z_t be an (p, q)-α^m-gf-closed set, pϵI_{p0} and $q \in I_{q1}$. Then $\alpha^m C_{\tau,\tau^*}(z_t, p, q) \leq z_{t_0}$, and hence $\alpha^m C_{\tau,\tau^*}(z_t, p, q)(x) \leq t_0$ also, for each x ∈, $z_t \setminus \{z_{t0}\}$ so, $\alpha^m C_{\tau,\tau^*}(z_t, p, q)(x) = 0$. Thus $\alpha^m C_{\tau,\tau^*}(z_t, p, q)$ is a fuzzy point. Therefore, z_t is an (p, q)-fuzzy just -α^m-closed.

Definition 4.16. Let (X, τ, τ^*) be a dfts. A fuzzy set λ of I^x is said to be an (p, q)-fuzzy α^m-nearly crisp if $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \wedge \alpha^m I_{\tau,\tau^*}(1-\lambda, p, q) = 0$, whenever p$\epsilon I_{p0}$ and $q \in I_{q1}$.

Theorem 4.17. If λ is an (r, s)-α^m-gf- closed and an (r, s)-fuzzyα^m-nearly crisp of a dfts (X, τ, τ^*), then $\alpha^m C_{\tau,\tau^*}(\lambda, p, q)\_\lambda$ does not contain any nonzero (p, q)-fα^m-closed set in X.

Proof: Assume that λ is an (p, q)-α^m-gf- closed set in I^x and μ is an (p, q)-fα^m-closed set such that $\mu \leq \alpha^m C_{\tau,\tau^*}(\lambda, p, q)\_\lambda$, and $\mu \neq 0$, whenever pϵI_{p0} and $q \in I_{q1}$. Then $\lambda \leq 1 - \mu$ and $1-\mu$ is an (p, q)-fα^m-open set. Since λ is an (p, q)-α^m-gf- closed set, then $\alpha^m C_{\tau,\tau^*}(\lambda, p, q) \leq 1-\mu$, hence

$$\mu \leq 1 - \alpha^m C_{\tau,\tau^*}(\lambda, p, q) = \alpha^m I_{\tau,\tau^*}(1-\lambda, p, q)$$

Then, $\mu \leq \alpha^m I_{\tau,\tau^*}(1-\lambda, p, q)$ But,

$$\mu \leq \alpha^m C_{\tau,\tau^*}(\lambda, p, q) \wedge \alpha^m I_{\tau,\tau^*}(1-\lambda, p, q) = 0.$$

Therefore $\mu = 0$, which is contradiction. Then $\alpha^m C_{\tau,\tau^*}(\lambda, p, q)\_\lambda$ does not contain any nonzero (p, q)-fα^m-closed set in X

Theorem 4.18. If λ is an (p, q)-α^m-gf-open and an (p, q)-fuzzyα^m– nearly crisp of adfts (X, τ, τ^*), then $\mu = 1$, $where$ μ is an (p, q)-fα^m-open and $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \vee (1-\lambda) \leq \mu$, for each $\lambda \in I^x$, pϵI_{p0} and $q \in I_{q1}$

Proof: let λ be any (p, q)-α^m-gf-$open$ set in X and μ be an (p, q)-fα^m-open set such that $\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \vee (1-\lambda) \leq \mu$. Then,

$$1- \mu \leq 1 - (\alpha^m I_{\tau,\tau^*}(\lambda, p, q) \vee (1-\lambda)) = 1 - \alpha^m I_{\tau,\tau^*}(\lambda, p, q) \wedge \lambda.$$

That is, $(1-\mu) \leq 1 - (\alpha^m I_{\tau,\tau^*}(\lambda, p, q)\_(1-\lambda)$

But, (1-μ) $\leq \alpha^{m}C_{\tau,\tau^*}(1-\lambda, p, q)\_(1-\lambda)$.

Since, 1- μ is an (p, q)-α^m-gf-closed set and $1-\lambda$ is an (p, q)$-\alpha^m$-gf-closed so, by Theorem 4.17, we have 1- $\mu = 0$ *then*, $\mu = 1$.

Conclusion

Our goal in this article is to introduce the notion of (p, q)-fuzzy α^m-closed sets, used this new idea to explain the notions (p, q)-α^m-generalized fuzzy-closed set in double fuzzy topological spaces. Also, we give some characterizations of these new terms with compared. Since double fuzzy topological forms is an extension of fuzzy topology and closeness is one of the basic notions for our research and our spaces which are widely used in the research field of machine learning, so we believe that our work and results can be useful to applied in
modern physics.

References

1. N. Levine, 1970,Generalized closed sets in topology, Rend. Circ. Mat. Palermo,19, 89-96.
2. J. Chang , 1968, Fuzzy topological spaces, Math . Anal. Appl. 20,182 − 189 .
3. L. A. Zadeh, Fuzzy sets, Information and control, 8, 1965, 338-353.
4. D. Coker and M. Dimirci, 1997, An introduction to intutionistic fuzzy topological
5. spaces in Sostak sense, Buseful, 67, 67-76.
6. K. Atanassov, 1987, Intuitionistic fuzzy sets, Fuzzy sets and system 20 (1), 87-96.
7. *J*. *G*. Garcia and S. E Rodabaugh, 2005, Order−theoretic, Topological, Categorical redundancides of interval-valued sets, Grey sets, Vague sets, Interval-valued " intuitionistic " sets, " intuitionistic" fuzzy sets and topologies, Fuzzy sets and system, 156, 445-484.
8. S. E. Abbas and Halis Aygun, 2006, Intuitionistic fuzzy semiregularization spaces, Information Science, 176, 745-757.
9. S. K. **S**amanta and T. K. Mondal, 2002, On intuitionistic gradation of openness , Fuzzy
10. sets , and Systems, 131,323-336.
11. F. M. Mohammed M.S.M. Noorani and A. Ghareeb, 2013, Generalized psi rho operations on double fuzzy topological spaces, Proceedings of the Universiti Kebangsaan Malaysia, Faculty of Science and Technology, Postgraduate Colloquium.

12. F. M. Mohammed M.S.M. Noorani and A. Ghareeb, 2014, Generalized Ψρ-closed sets and generalized Ψρ-open sets in double fuzzy topological spaces, AIP Conference Proceedings 1602(1):909-917.

13. M. Mathew, R. Parimelazhagan, 2014, αm –closed sets in topological spaces, Internat. J. Math. Analysis, 8, 2325-2329.

14. E. P. Lee and Y. B. Im, 2001, Mated fuzzy topological spaces, Int. Jfuzzy logic Intell.

15. Syst 11, 161 − 165.

16. Y. C. Kim and S. E. Abbas, 2004, Several types of fuzzy regular spaces, IndinanJ. Pure

17. Appl. Math. , 35,481 − 500 .

18. H. Gurcay, Es. A. Haydar and D. Coker, 1997, On fuzzy continuity in intuitionistic fuzzy topological spaces, J.Fuzzy Math.5 (2), 365-378.

19. A. Ghareeb, 2011, Normality in double fuzzy topological spaces, Applied Mathematical Letters, 24 ,533 − 540.

20. A D. Kalamani, K. Sakthivel and C.S. Gowri, 2012, Generalized alpha closed sets in intuitionistic fuzzy topological spaces, Applied Mathematical Sciences, 6(94), 4691-4700.

21. S. E. Abbas E. El-Sanosy, 2012, Characterizations of some double fuzzy separation axioms, The Journal of fuzzy mathematics 20, 47-62.