On Definition of Expansion Probabilistic Hilbert Space

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Received: 12 September 2017 Accepted: 19 December 2017

Abstract

The purpose of this paper is to give a new definition for Expansion probabilistic inner product space (EPIP-space) is given. Based on this definition several convergence theorems and definition for (EPH-space) are instituted and introduced. More over several linear bounded operators are given with some properties of such operators studied in this paper.

Keyword: linear bounded operator, EPH-space, τ-convergent, EPIP-space, self-adjoint

حول تعريف فضاء هلبرت الاحتمالي الموسع

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الخلاصة

الهدف من البحث اعطاء تعريف جديد لفضاء هلبرت الاحتمالي الموسع (فضاءEPIP) وبالاعتماد على هذا التعريف قدمنا مجموعه من مبرهنات التقارب وعرفنا فضاء هلبرت الاحتمالي الموسع(فضاء EPH). كذلك تم تعريف مجموعه من المؤثرات الخطيه المحدوده وبرهان النظريات المتعلقه بها.

الكلمات المفتاحية: المؤثر الخطي المحدود، فضاء EPH، تقارب -t، فضاء EPIP، الترافق الذاتي.

DIYALA JOURNAL FOR PURE SCIENCES

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Introduction

In 1994, S.S Chang, proposed the definition of a probabilistic inner-product space (PIP-space) [1]. In 2001 Yongfu Su. inserted a modification on this Chang$s$ definition [2]. In 2007 Yongfu et al. introducing the definition of Probabilistic Hilbert Space (PH-space) [3]. In 2014 Radhi I. M. Ali et al. defined the adjoint operator on PH- Space [4]. In 2015 Radhi I.M. Ali et al. introduced the certain types of bounded operator on PH-space, also in proved that any operator on PH-space is self-adjoint operator [5]. A self-adjoint operator has several applications in the field of quantum machines [6,7]. The aim of this paper is introducing a new definition for the EPH- space. Based on the proposed definition, we concluded several basic properties results for this type of spaces such as boundedness, convergence and the relationship between bounded operator on EPH- space and PH- space. This paper is organization as follows: section I is the introduction of the paper, section II shows the preliminaries, section III presents the PH- space and section IV presents the EPH-space.

1. Preliminaries

In this section we introduce several important definition and theorems of a PIP –space. Theorems in this section are stated without proof, and we will rely sources [4], [5], [8].

Definition 1.1

A function G from $\bar{R} = [-\infty, +\infty]$ into I=[0,1] is called a distribution function (df), that is non-decreasing and left continuous with $inf_{\ t\in R}$ G(t) = 0 , $sup_{t\in R}\, G(t) = 1$

Definition 1.2

A PIP –space is a triple (S, G ,⋆) where S is a real linear space and G function from S×S into D (D: set of all df´s)

is denoted by $G_{x,y}(t)$ for each (x, y) ∈ S×S satisfies the following:

(PIP-1) $G_{x,x}(0) = 0$ $, for\ all\ x\ belong\ to\ S$

(PIP-2) $G_{x,y}(t) = G_{y,x}(t)\,.\, for\ all\ x\ and\ y\ belong\ to\ S$

(PIP-3) $G_{x,x}(t) = H(t)\ iff x = 0$

Where $H(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ 1 & \text{if} \quad t > 0 \end{cases}$

(PIP – 4) For x and y belong to S and ∝ real number

$$G_{\propto x,y}(t) = \begin{cases} G_{x,y}\left(\frac{t}{\propto}\right) & \propto > 0 \\ H\,(\,t) & \propto = 0 \\ 1 - G_{x,y}\left(\frac{t}{\propto} +\right) & \propto < 0 \end{cases}$$

Where $G_{x,y}(t +) = \lim_{t \to t^{+}} G_{x,y}(t^{1})$

(PIP- 5) If x with y is linearly independent then $G_{x+y,z}(t) = (G_{x,z} * G_{z,y})(t)$

Where $((G_{x,z} * G_{z,y})(t) = \int_{-\infty}^{+\infty} G_{x,z}\,(t-u) G_{z,y}(u)$

Definition 1.3:

Assume that (S, G, ⋆) be a PIP space, if $\int_{-\infty}^{+\infty} td\ G_{x,y}(t)$

is convergent for all x, y ∈ S.

Then (S, G, ⋆) is called PIP- space with mathematical exception (ME)

Theorem 1.1:

Assume that (S, G, ⋆) be a PIP space with ME. If

$< x,y> =\int_{-\infty}^{+\infty} td\ G_{x,y}(t) \quad , \quad \forall\ x,y\ \in S$

Then (S, G, ⋆) is inner – product space, so that (S, ||.||), is a normal space where ‖x‖ = $\sqrt{< x,x >}$, $for\ all\ x\ belong\ to\ S$

Definition 1.4

Let (S, G, ⋆) be a PIP space with ME, then if S is completed in ‖. ‖ then is called PH-space.

2. Probabilistic Hilbert Space

This section consists of some important definitions and theorems of a PH- space, in this section we will rely sources, [4] and [5].

Theorem 2.1: (Riesz theorem)

Let (S, G, ⋆) be a PIP space, for any Linear continues function g(x), ∃ unique y ∈ S such that

g(x)= $\int_{-\infty}^{+\infty} td\ G_{x,y}(t) \quad , \quad \forall\ x \in s$

Definition 2.1:

Let (S,G,⋆) be a PIP space , let L be a linear operator defined on normed space S. Then L is said to bounded in norm if

∃ a constant M>0 S.t

|‖LX||≤ L ||x||, ∀ x ∈S

Definition 2.2:

Let (S,G,⋆) be a PH-space with ME and let L be a Linear operator on S Then L is said to be F-bounded operator if there exist a constant K>0 , S.t

$G_{Lx,LX}(t) \geq G_{x,x}\left(\frac{t}{k}\right) \quad , \quad for\ all\ x \in S$

Theorem 2.2:

Let (S, G, ⋆) be a PH-space with ME. Let L be a Linear operator defined on S, then L is F-bounded iff L is bounded in norm on S.

Theorem 2.3:

Let (S, G, ⋆) be a PH-space, Let L be a continuous Linear operator defined on S then ∃ unique $L^{\star} \in$ (S, G, ⋆) linear operator satisfying:

$< L x, y> = < x, L^{\star} y > \ , \ \forall x, y \in s$

$L^{\star}$ is Adjoint operator.

Definition 2.3:

Let (S, G, ⋆) be a PH-space with ME, Let L be a Linear operator defined on S then L is self - adjoint operator if $L=L^{*}$.

Remark 2.1:

1.Since S is real linear space, and $< Lx,x> = \int_{-\infty}^{+\infty} td\, G_{Lx,x}(t) < \infty$ Then < Lx, x> real for any x ∈S, L be a linear operator on S .
2.L is self – adjoint operator.
3.(S, G, ⋆) is a real Hilbert Space.

3. Expansion probabilistic Hilbert Space

This section consists a new definitions and theorems about a EPIP-space, EPH-space and bounded linear operators on EPH-space.

Definition 3.1:

A EPIP- space is a triple (S+iS, G, ⋆) where:

S+ iS = {x+ iy : x, y ∈ S}, and G a function from

S to D(as defined in the definition 2.2)

Theorem 3.1:

Let (S+iS ,G,⋆) be a EPIP –space ,letting :

$$< x+iy / u+iv > = \int_{-\infty}^{+\infty} td\ G_{x,u}(t) + \int_{-\infty}^{+\infty} td\ G_{y,v}(t) + i\left(\int_{-\infty}^{+\infty} td\ G_{x,v}(t) - \int_{-\infty}^{+\infty} td\ G_{y,u}(t)\right).$$

$$= <x,u> + <y,v> + i(<x,v> - <y,u>)$$

For all (x+iy) and (u+iv) ∈ S+iS, where $<.,.>$ the inner –product defined on S (Theorem 2.4). Then (S+iS, $</.>$) is a EIP – space, so that (S+iS,‖.‖) is a Expansion normed space (EN-space),

Where $\|x+iy\|=\sqrt{< x,x > + < y,y >}$

For all (x+iy) ∈ S+iS.

Proof:

Respectively, we verify the conditions of inner product. we have:

1. $<x+iy/x+iy> = \int_{-\infty}^{+\infty} td\ G_{x,x}(t) + \int_{-\infty}^{+\infty} td\ G_{y,y}(t) + i(\int_{-\infty}^{+\infty} td\ G_{x,y}(t) - \int_{-\infty}^{+\infty} td\ G_{y,x}(t))$

$$=<x,x>+<y,y>+i(<x,y> -<y,x>)$$

By theorem 2.4 we get:

$<x+iy/x+iy>=<x,x>+<y,y> \geq 0$

For all (x+iy) ∈ S+iS

2. $< x+iy/x+iy> = < x,x>+<y,y>$ (from1)

$$= 0$$

If and only if x=0 and y=0 for all x+iy ∈ S+iS

3. $< x+iy/u+iv> = \int_{-\infty}^{+\infty} td\ G_{x,u}(t) + \int_{-\infty}^{+\infty} td\ G_{y,v}(t) + i(\int_{-\infty}^{+\infty} td\ G_{x,v}(t) - \int_{-\infty}^{+\infty} td\ G_{y,u}(t))$

By definition 2.2 we get :

$< x+iy / u+iv> = \int_{-\infty}^{+\infty} td\ G_{u,x}(t) + \int_{-\infty}^{+\infty} td\ G_{v,y}(t) + i(\int_{-\infty}^{+\infty} td\ G_{v,x}(t) - \int_{-\infty}^{+\infty} td\ G_{u,y}(t))$

$= \overline{< u + iv/x + iy >}$

for all (x+iy) and (u+iv) ∈ S+iS

4. For all (x+iy) and (u+iv) ∈ S+iS , and ∝ is a scalar we have :

< ∝(x+iy) / (u+iv) > = < ∝x+i∝y/ u+iv >

$= \int_{-\infty}^{+\infty} td\ G_{\propto x,u}(t) + \int_{-\infty}^{+\infty} td\ G_{\propto y,v}(t) + i(\int_{-\infty}^{+\infty} td\ G_{\propto x,v}(t) - \int_{-\infty}^{+\infty} td\ G_{\propto y,u}(t))$

by the definition 2.2 and the theorem 2.4 we get :

< ∝(x+iy) /(u+iv)> = ∝<x,u>+∝<y,v>+i(∝<x,v>-∝<y,u>)

= ∝ (< x,u > + < y,v > + i(< x,v > - < y,u >))

= ∝ < x+iy/ u+iv>

5. For all (x+iy) , (u+iv) and (w+iz) ∈ S+iS

< (x+iy)+(u+iv) /(w+iz)> = <(x+u) + i(y+v)/w+iz>)

$= \int_{-\infty}^{+\infty} td\ G_{x+u,w}(t) + \int_{-\infty}^{+\infty} td\ G_{y+v,z}(t) + i(\int_{-\infty}^{+\infty} td\ G_{x+u,z}(t) - \int_{-\infty}^{+\infty} td\ G_{y+v,w}(t))$

= < x+u,w>+<y+v,z> + i(<x+u,z>-<y+v,w>)

By theorem 2.4 we get:

< (x+iy) +(u+iv)/w+iz> =< x,w>+< y,z> + i(<x,z>-<y,w>)+<u,w>+<v,z>+i(<u,z> - <v,w>)

=< x+iy/w+iz+< u+iv/w+iz>

Definition 3.2:

Let (S+iS,G,⋆) be a EPIP –space ,then :

1. A sequence $(\boldsymbol{x_n} + \boldsymbol{iy_n})$ in S+iS is say to be T- convergent to (x+iy) ∈ S+iS, if for every ∈>0 and $\boldsymbol{\lambda}$>0 there exist positive integer N=N (∈, $\boldsymbol{\lambda}$) s.t:

$$G_{x_n - x, x_n - x} + G_{y_n - y, y_n - y}(\in) > 1 - \lambda \ , \forall\, n > N$$

2. A sequence $(x_n + \boldsymbol{iy_n})$ in S+iS is say to be T- Cauchy sequence, if for every ∈>0 and $\boldsymbol{\lambda}$>0 there exist positive integer N=N (∈, $\boldsymbol{\lambda}$), such that:

$$G_{x_n - x_m, x_n - x_m}(\in) + G_{y_n - y_m, y_n - y_m}(\in) > 1 - \lambda \ , \forall\, n, m > N$$

3. (S+iS,G,⋆) is say to be T- complete if each T- Cauchy sequence in S+iS is T- convergent in S+iS .

4. A linear function f(x+iy) defined in S+S is called continuous , if for any sequence $(\boldsymbol{x_n} + \boldsymbol{iy_n})$ in S+iS that T- converges to x+iy ∈ S+iS then f$(\boldsymbol{x_n} + \boldsymbol{iy_n})$ ⇁ f(x+iy)

5. If S+iS is complete in the ||.|| then S+iS is called Expansion probabilistic Hilbert Space (EPH- space), where ||x+iy|| = $\sqrt{< x, x > + < y, y >}$ for all x+iy ∈ S+iS .

Theorem 3.2:

Let (S+iS, G, ⋆) be a EPIP – space . For sequence ($\boldsymbol{x_n} + \boldsymbol{iy}_{\boldsymbol{n}}$) in S+iS, m –convergent (in norm, $||.||^2$= <./.> = <.,.> +<.,.>) implies T- convergent.

Proof:

Let ($\boldsymbol{x_n} + \boldsymbol{iy}_{\boldsymbol{n}}$) m – convergent to a point (x+iy), then

$$\lim_{n\to\infty} < (x_n + iy_n) - (x + iy) \,/\, (x_n + iy_n) - (x + iy) >$$

$= \lim_{n\to\infty} < (x_n - x) + i(y_n - y) / (x_n - x) + i(y_n - y) > = 0$

Which leads to

$$\lim_{n\to\infty} \left(\int_0^{+\infty} td\ G_{x_n - x, x_n - x}(t) + \int_0^{+\infty} td\ G_{y_n - y, y_n - y}(t) + i\left(\int_0^{+\infty} td\ G_{x_n - x, y_n - y}(t) - \int_0^{+\infty} td\ G_{y_n - y, x_n - x}(t) \right) \right) = 0$$

By definition 2.2 we get:

$$\lim_{n\to\infty} \left(\int_0^{+\infty} tdG_{x_n - x, x_n - x}(t) + \int_0^{+\infty} tdG_{y_n - y, y_n - y}(t) \right) = 0 \qquad (4-1)$$

Hence for any $\in > 0$, $0 < \boldsymbol{\lambda} < 1$, observe:

$$\int_0^{+\infty} tdG_{x_n - x, x_n - x}(t) + \int_0^{+\infty} tdG_{y_n - y, y_n - y}(t)$$

$$= \int_0^{\in} tdG_{x_n - x, x_n - x}(t) + \int_{\in}^{+\infty} tdG_{x_n - x, x_n - x}(t) + \int_0^{\in} tdG_{y_n - y, y_n - y}(t) + \int_{\in}^{+\infty} tdG_{y_n - y, y_n - y}(t)$$

$$= \int_0^{\in} td(G_{x_n - x, x_n - x} + G_{y_n - y, y_n - y})(t) + \int_{\in}^{+\infty} td(G_{x_n - x, x_n - x} + G_{y_n - y, y_n - y})(t)$$

(4-2)

$$\geq \int_{\in}^{+\infty} \in dt(G_{x_n - x, x_n - x} + G_{y_n - y, y_n - y})\ (t)$$

$= \in [1 - (G_{x_n - x, x_n - x} + G_{y_n - y, y_n - y})(\in)]$

It follows from (4-1) and (4-2) that, ∃ positive integer N, if n>N then

$(G_{x_n - x, x_n - x} + G_{y_n - y, y_n - y})\ (\in) > 1 - \lambda$

So that ($x_n + iy_n$) T- converges t x+iy .

Definition 3.3:

Let (S, G, ⋆) is a PH-space, and (S+iS, G, ⋆) be a EPH-space, let L is a linear operator on S. Defined T is a linear operator on S+iS as: for all (x+iy) ∈ S+iS

T(x+iy) = Lx+iLy

Definition 3.4:

Let T de a linear operator defined on EPH-space S+iS, then T is say to be bounded in norm if: ∃ M >0 s.t:

$\|T(x+iy)\| \leq M \|x+iy\|$, ∀(x+iy) ∈ S+iS

Theorem 4.3:

Let L to be a linear operator on PH-space S and T be a linear operator on EPH-space S+iS, s.t:

T(x+iy) = Lx+iLy, ∀ (x+iy) ∈ S+iS

Then T is bounded if and only if L is bounded.

Proof:

Since L is bounded operator on S, then ∃ M > 0, s.t:

$\|Lx\| \leq M \|x\|$, ∀ x∈S

$\| T(x+iy)\|^2 = \|Lx+iLy\|^2$

$= < Lx + iLy/Lx + iLy >$

$= \int_{-\infty}^{+\infty} tdG_{Lx,Lx}(t) + \int_{-\infty}^{+\infty} tdG_{Ly,Ly}(t) + i(\int_{-\infty}^{+\infty} tdG_{Lx,Ly}(t) - \int_{-\infty}^{+\infty} tdG_{Ly,Lx}(t))$

theorem 2.4 we get : By

$\| T(x+iy)\|^2$ = < Lx,Lx> + <Ly,Ly>

$= \|Lx\|^2 + \|Ly\|^2$

$\leq M^2\|x\|^2 + M^2\|y\|^2$

$= M^2(\|x\|^2 + \|y\|^2)$

$= M^2\|x + iy\|^2$

Then $\| T(x+iy) \| \leq M \|x+iy\|$

T is bounded operator.

Let T is bounded operator on S+iS , then ∃ k > 0 , such that

$\| T(x+iy)\| \leq k\| x+iy\|$, ∀ x+iy ∈ S+iS .

$\| T(x+iy)\|^2 \leq k^2\|x + iy\|^2$

$= k^2(\|x\|^2 + \|y\|^2)$

$= k^2\|x\|^2 + k^2\|y\|^2$ (4-3)

$\| T(x+iy) \|^2 = \| Lx+iLy\|^2$

$$= \int_{-\infty}^{+\infty} t dG_{Lx,Lx}(t) + \int_{-\infty}^{+\infty} t dG_{Ly,Ly}(t)$$

$$+ i(\int_{-\infty}^{+\infty} t dG_{Lx,Ly}(t) - \int_{-\infty}^{+\infty} t dG_{Ly,Lx}(t)$$

$= \langle Lx,Lx\rangle + \langle Ly+Ly\rangle$

$= \|x\|^2 + \|y\|^2$ (4-4)

From (4-3) and (4-4), we get

$\|Lx\|^2 + \|Ly\|^2 \leq k^2\|x\|^2 + k^2\|y\|^2$

Then L is bounded operator.

Definition 3.5:

Let (S+iS, G, ⋆) be a EPH – space, then:

1. A linear operator C on S+iS is say to be complex conjugation operator (CC– operator) if :
   C(x+iy) = x-iy, ∀ (x+iy) ∈ S+iS
2. A linear operator P on S+iS is say to be unitary flip operator (UF- operator) if :
   P(x+iy) = y-ix, ∀ x+iy ∈ S+iS
3. A bounded Linear operator T on S+iS is say to be real If :

   CT=TC

Remark 3.1:

1. The UF – operator P satisfies :
   $P = P^{\star} = P^{-1}$
2. CC = I

Theorem 3.4:

Let T be a Linear operator definition on EPH – space . Then T is say to be bounded and real if there is a (uniquely determined) pair of bounded operator $L_j: S \rightarrow S$, j= 1,2 s.t : T(x+iy) = $L_1x + iL_2y$, $\forall x + iy \in S + iS$

Proof:

Let $L_j: S \rightarrow S$, j= 1, 2 bounded operators and

T (x+iy) = $L_1x + iL_2y$, $\forall x + iy \in S + iS$

T $C(x + iy)$ = T (x-iy)

$= L_1x + L_2y$

$= C(L_1x + iL_2y)$

$= CT(x + iy)$

Then TC =TC (4-5)

Since L_1 and L_2 bounded operator ∃ $M_1, M_2 > 0$ s.t :

$\|L_1x\| \leq M_1 \|x\|$

$\|L_2y\| \leq M_2 \|y\|$

For all x,y ∈ S

$\| T(x+iy) \|^2 = \|L_1x + iL_2y\|^2$

$= < L_1x , L_1x > + < L_2y , L_2y >$

$= \| L_1x\|^2 + \| L_2y \|^2$

$\leq M_1\|x\|^2 + M_2\|y\|^2$

$\leq M (\|x\|^2 + \|y\|^2)$

Where $M^2 = \text{Max}\,({M_1}^2, {M_2}^2)$

$\| T(x+iy) \|^2 \leq M^2(\|x\|^2 + \|y\|^2)$

$= M^2\|x+iy\|^2$

$\| T(x+iy)\| \leq M\| x+iy\|$ (4-6)

Then T is bounded operator from (4-5) and (4-6) we get: The operator T is bounded and TC = CT

Then T is real (definition 4.8)

Theorem 3.5:

Let T be a Linear operator defined on EPH – space. Then T is say to be real and TP=PT, if there is

a (uniquely determined) bounded operator $L: S \to S$ s.t: $T(x+iy) = Lx + iLy$, for all $x + iy \in S + iS$

Proof:

Let $L: S \to S$ be a linear bounded operator s.t:

$T(x + iy) = Lx + iLy$ for all $x + iy \in S + iS$

From theorem (4.10) we get: T is real

$$PT(x + iy) = P(Lx + iLy)$$
$$= Ly - iLx$$
$$= T(y - ix)$$
$$= TP(x + iy)$$

Then PT=TP

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