A study the Spectrum for A Probabilistic Hilbert Space Operator

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Received: 5 September 2016 **Accepted: 31 October 2016**

Abstract

The objective of this work is to exhibit an integrated to the spectrum of bounded linear operator in probabilistic Hilbert space (PH-space). We have given several theorems, and some of its essential properties to the spectrum of the bounded linear operators in PH-space.

Keywords: spectrum, PH-space , bounded operator , self-adjoint .

دراسة الطيف لمؤثرات فضاء هلبرت الاحتمالي

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الخلاصة

الهدف من هذا العمل عرض نهج متكامل الى اطياف المؤثرات الخطية المقيدة في فضاء هلبرت الاحتمالي . لقد اثبتنا عدة مبرهنات وكذلك اعطينا بعض الخصائص الاساسية للمؤثرات الخطية المقيدة في فضاء هلبرت الاحتمالي .

الكلمات المفتاحية : الطيف ، فضاء هلبرت الاحتمالي ،المؤثر المقيد ، مترافق ذاتيا.

Introduction

The definition of on the probabilistic inner product spaces (PIP-space) insert by S.S.chang [1] , Yongfu su .[2] , modified the S.S.changs definition .The PIP-space $(E, G,*)$ where E is a real linear space and mapping $G: E \times E \to D$ (D: set of all distribution function) is denoted by $G_{u\times v}(x)$ for each $(u, v) \in E \times E$ achieve the following conditions :

1. $G_{u\times v}(0) = 0$, $\forall\, u \in E$;

2. $G_{u\times v}(x) = G_{u\times v}(x)$, $\forall\, u, v \in E$;

3. $G_{u\times v}(x) = H(x)$ if and only if $u = 0$

Where $H(x) = \begin{cases} 0 & if\ \ x \le 0 \\ 1 & if\ \ x > 0 \end{cases}$

4. $G_{u\times v}(x) = \begin{cases} G_{u\times v}\left(\frac{x}{\lambda}\right) & \lambda > 0 \\ H(x) & \lambda > 0 \\ 1 - G_{u\times v}\left(\frac{x}{\lambda} + 1\right) & \lambda > 0 \end{cases}$

Where $G_{u\times v}(x) = \lim_{x' \to x^+} G_{u\times v}(x)$

5. If u with v is linearly independent then

$$G_{u+v,w}(x) = \left(G_{u,w} * G_{w,v}\right)(x)$$

Where

$$\left(G_{u,w} * G_{w,v}\right)(x) = \int_{+\infty}^{-\infty} G_{u,w}(x-t)\, dG_{w,v}(t)$$

A PIP-space is called mathematical assumption if :

$$\int_{-\infty}^{+\infty} x\, dG_{u,v}(x) \;\; is\ convergent\ for\ each\ u, v \in E \;\; .if:$$

$$< u, v > = \int_{-\infty}^{+\infty} x\, dG_{u,v}(x)\ for\ each\ u, v \in E$$

Then $(E, <\cdot,\cdot>)$ is PH-space so that $(E, \|\cdot\|)$ is a normal space , where $\| u \| = \sqrt{< u, v >}$ for each $u \in E$.

If E is complete in the $\|\cdot\|$, then E is called PH-space ,[3]. Let the set of all bounded operators acting on (E) .

Operators on ph-space

The private set of operators on ph-space is known as follows: [4]

1. S is self-adjoint if

$$S^* = S$$

On a par with

$$\int_{-\infty}^{+\infty} x\, dG_{Su,u}(x) \;\; belong\ to\ real\ numbers\ for\ each\ u \in E$$

2. S is positive if:

$$< Su, v > = \int_{-\infty}^{+\infty} x\, dG_{Su,u}(x) \;\; \geq 0 \quad , \quad for\ each\ u \in E$$

3. S is normal if :

$$S^*S = SS^*$$

On a par with

$$\int_{-\infty}^{+\infty} x\, dG_{S^*Su,u}(x) = \int_{-\infty}^{+\infty} x\, dG_{SS^*u,u}(x) \quad for\ each\ u \in E$$

4. S is unitary operator if:

$$SS^* = I = S^*S$$

On a par with

$$\int_{-\infty}^{+\infty} x\, dG_{SS^*u,U}(x) = \int_{-\infty}^{+\infty} x\, dG_{Iu,u}(x) = \int_{-\infty}^{+\infty} x\, dG_{S^*Su,U}(x)$$
$$for\ each\ u \in E$$

5. S is isometric operator if:

$$S^*S = I$$

On par with

$$\int_{-\infty}^{+\infty} x\, dG_{S^*Su,U}(x) = \int_{-\infty}^{+\infty} x\, dG_{Iu,u}(x) = for\ each\ u \in E$$

Spectrum of Operator

In this part insert the spectrum for operator $S \in B(E)$: where B(E)is all bounded operators acting on E

1. Resolving set of :

$$\rho(S) = \{\lambda : S - \lambda I \text{ is invertible } \}$$

And $\rho(S)$ *is open subset of* $\mathbb{R}$.

2. The spectrum of :

$$\sigma(S) = \mathbb{R} \backslash \rho(S)$$

And $\sigma(S)$ is closed.

3. The point spectrum of :

$$\sigma_\rho(S) = \{\lambda : S - \lambda I \text{ is not one} - \text{to} - \text{one } \}$$

4. The continuous spectrum of :

$$\sigma_\rho(S) = \begin{cases} \lambda : S - \lambda I \text{ is one} - \text{to} - \text{one, and} \\ R(S - \lambda I) \text{is a proper dence subspace of } E \end{cases}$$

5. The Residual spectrum of S :

$$\sigma_r(S) = \begin{cases} \lambda : S - \lambda I & \text{is not one} - \text{to} - \text{one , and} \\ \overline{R(S - \lambda I)} & \text{is a proper subspace of } E \end{cases}$$

6. The approximate point spectrum of S :

$$\sigma_{ab}(S) = \begin{Bmatrix} \lambda : \forall\, \epsilon > 0 \,, \exists\, u \in E \,, \text{unit vector} \\ \text{such that } \parallel Su - \lambda u \parallel < \epsilon \end{Bmatrix}$$

4. Main results

We have given several theorems, and some of essential properties to the spectrum of the operator $\in B(E)$.

Lemma 1:

Let $S \in B(E)$ is normal, then:

i. If $Su = \lambda u$ for some $\lambda \in \mathbb{R}$ and $u \in E$,then

$$S^{*}u = \lambda u$$

ii. If $\lambda_1 \neq \lambda_2$ ($\lambda_1 , \lambda_2 \in \mathbb{R}$) , then

$$\ker(S - \lambda_1 I) \perp \ker(S - \lambda_2 I)$$

Proof:

(i) By normality of , for each $u \in E$

$$\| (S - \lambda I)u \|^2 = \| (S - \lambda I)^{*}u \|^2$$

$$\int_{-\infty}^{+\infty} x \, dG_{(S-\lambda I)u,(S-\lambda I)u}(x) = \int_{-\infty}^{+\infty} x \, dG_{(S-\lambda I)^{*}u,(S-\lambda I)^{*}u}(x)$$

It implies (i).

(ii) Suppose that

$u, v \in E$ and $\lambda_1 \neq \lambda_2$ are in $\mathbb{R}$ such that

$Su - \lambda_1 u$ and $Su - \lambda_2 u$ then

$$\lambda_1 < u, v >= \lambda_1 \int_{-\infty}^{+\infty} x \, dG_{u,v}(x)$$

$$= \int_{-\infty}^{+\infty} x \, dG_{\lambda_1 u,v}(x)$$

$$= \int_{-\infty}^{+\infty} x \, dG_{Su,v}(x)$$

$$= \int_{-\infty}^{+\infty} x \, dG_{u,S^*v}(x)$$

$$= \int_{-\infty}^{+\infty} x \, dG_{u,\lambda_2 v}(x)$$

$$= \lambda_2 \int_{-\infty}^{+\infty} x \, dG_{u,v}(x)$$

$$= \lambda_2 < u, v >$$

$\Longrightarrow$

$$\lambda_1 < u, v >= \lambda_2 < u, v >$$

$$(\lambda_1 - \lambda_1) < u, v >= 0$$

$$since \ \lambda_1 \neq \lambda_2 \quad then \ < u, v >= 0$$

Then

$$\ker(S - \lambda_1 I) \perp \ker(S - \lambda_2 I)$$

Remark:

1. $S_\lambda = S - \lambda I$

2. $\sigma_\rho(S) \cup \sigma_C(S) \subset \sigma_{ab}(S) \subset \sigma(S)$

3. If S is normal then:

   i. $\sigma_\rho(S) \cup \sigma_c(S) = \sigma_s = \sigma_{ab}(S)$

   ii. $\sigma(S) = \{\lambda \in \mathbb{R} : R(S_\lambda) = E\}$

   iii. $\sigma_\rho(S) = \{\lambda \in \mathbb{R} : \overline{R(S_\lambda)} \neq E$

vi. $\sigma_c(S) = \{\lambda \in \mathbb{R} : \overline{R(S_\lambda)} = E \text{ } and \text{ } R(S_\lambda) \neq E \}$

v. $\sigma_r(S)$ is empty

4. $\sigma_{ap}(S)$ is defined as: $\forall \lambda \in \mathbb{R} \, \exists U_n \in E \ (Uint \ vectors) \ such \ that$

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} x \, dG_{S_\lambda U_n, S_\lambda U_n}(x) = 0$$

5. Let $S \in B(E)$ and $U_n \in E$, U_n called S-spectral if :

i. U_n unit vectors

ii. $\lim_{n\to\infty} \int_{-\infty}^{+\infty} x \, dG_{SU_n, SU_n}(x) = 0$

Theorem 2 :

Let $(E, G, *)$ be a PH-space , let $S \in B(E)$ and $\lambda \in \mathbb{R}$. And let $U_n \in \mathbb{R}$ be a λ-sequence which is not weakly converging to zero ($U_n \nrightarrow^w 0$) , then $\lambda \in \sigma_p(S)$.

Proof:

Since $\lambda \in \mathbb{R}$ and ($U_n \nrightarrow^w 0$) , then $\exists t \neq 0 \in E$ and $V_k = U_n \subset U_n$ satisfies the following:

1. V_k unit vectors and

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} x \, dG_{S_\lambda V_k, S_\lambda V_k}(x) = 0$$

(i.e. V_k is S_λ-sequence)

2. $V_k \longrightarrow^w t \ as \ k \longrightarrow \infty$ By using B-saka theorem ([5] and [6] , p154) , $\exists \, t_m = V_k \subset V_k$ such that $\tilde{t}_m \longrightarrow t$ (converging strongly),

$$\text{where} \quad \tilde{t}_m = \frac{1}{m}\sum_{j=1}^{m} t_j = \frac{1}{m}\sum_{j=1}^{m} V_{kj}\,, \quad \forall\, m \geq 1$$

Since V_k is S_λ-sequence then by Cesaro's means. Converges theorem then:

$$\int_{-\infty}^{+\infty} x\, dG_{S_\lambda \tilde{t}_m, S_\lambda \tilde{t}_m}(x) = \frac{1}{m}\int_{-\infty}^{+\infty}\sum_{j=1}^{m} x\;dG_{S_\lambda V_{kj}, S_\lambda V_{kj}}(x)$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\int_{-\infty}^{+\infty} x\, dG_{S_\lambda V_k, S_\lambda V_k}(x) \longrightarrow 0 \quad , as\ k \longrightarrow \infty$$

Since S is continuous, we get

$$\int_{-\infty}^{+\infty} x\, dG_{S_\lambda t, S_\lambda t}(x) = \lim_{m\to\infty}\int_{-\infty}^{+\infty} x\, dG_{S_\lambda \tilde{t}_m, S_\lambda \tilde{t}_m}(x) = 0$$

Then $\in \sigma_p(S)$.

Theorem 3:

Let $(E, G, *)$ be a PH-space .let $S \in B(S)$ be a normal operator then there exists $\lambda \in \sigma(S)$ such that $|\lambda\,| = \parallel S \parallel$.

Proof:

Assume that $\neq 0$, since S is normal then

$$\parallel S \parallel = \sup_{\|u\|=1} |< Sx, x >|$$

$\exists U_n \in E$ unit vector such that:

$$\lim_{n\to\infty}\left|\int_{-\infty}^{+\infty} x\, dG_{SU_n,U_n}(x)\right| = \| S \|$$

$$\Longrightarrow \lim_{n\to\infty}\int_{-\infty}^{+\infty} x\, dG_{SU_n,U_n}(x) \qquad is\ convergent$$

Let λ be the limit of this sequence.

$$\Longrightarrow \lambda = \| S \|$$

To prove $\lambda \in \sigma(S)$,To prove U_n is S_λ-spectral sequence:

$$\int_{-\infty}^{+\infty} x\, dG_{S_\lambda U_n, S_\lambda U_n}(x) = \int_{-\infty}^{+\infty} x\, dG_{SU_n-\lambda U_n, SU_n-\lambda U_n}(x)$$

$$= \int_{-\infty}^{+\infty} x\, d\big(G_{SU_n,SU_n}(x) + G_{-\lambda U_n,SU_n}(x) + G_{SU_n,-\lambda U_n}(x) + G_{-\lambda U_n,-\lambda U_n}(x)\big)$$

$$= \int_{-\infty}^{+\infty} x\, dG_{SU_n,SU_n}(x) - 2\lambda \int_{-\infty}^{+\infty} x\, dG_{SU_n,U_n}(x) + \lambda^2 \int_{-\infty}^{+\infty} x\, dG_{U_n,U_n}(x)$$

$$= \| SU_n \|^2 - 2\lambda < SU_n, U_n > +\lambda^2 \| U \|$$

$$= \| SU_n \|^2 - 2\lambda < SU_n, U_n > +\lambda^2$$

$$\le 2\lambda^2 - 2\lambda < SU_n, U_n >$$

$$\longrightarrow 2\lambda^2 - 2\lambda^2 \longrightarrow 0 \quad as \quad n \longrightarrow \infty$$

Thus $\lambda \in \sigma(S)$.

Theorem 4:

Let $(E, G, *)$ be a PH-space, and let assume that $T_s(\lambda) = \emptyset$ and S_λ not one –to- one. Then $\in \sigma(S)$.

Proof:

Since $T_s(\lambda) = \emptyset$, then

$$\alpha : \inf_{u \in S(E)} \int_{-\infty}^{+\infty} x \, dG_{S_\lambda u, S_\lambda u}(x) > 0$$

$$where \;\; S(E) = \{u \in E : \| u \| = 1$$

We have

$$\int_{-\infty}^{+\infty} x \, dG_{S_\lambda u, S_\lambda u}(x) \geq \alpha \int_{-\infty}^{+\infty} x \, dG_{u,u}(x) \qquad , \forall \; u \in E$$

$\Longrightarrow S_\lambda$ is one-to-one and $R(S_\lambda)$ is closed in E .

Since S_λ is not one-to-one.

$\Longrightarrow R(S_\lambda)$ is not dense in E

$\Longrightarrow \lambda \in \sigma_r(S)$.

Theorem 5:

Let $(E, G, *)$ be a ph-space , and $S \in B(E)$ be a normal . let $\lambda \in \sigma(S)$ then $S_\lambda(E)$ is not closed .

Proof:

If S_λ is one-to-one and $S_\lambda(E)$ closed, then by the inverse mapping theorem, $\exists \, T$ continuous linear map :

$T : S_\lambda(E) \longrightarrow E$ such that

$T_\lambda u = u$ for each $u \in E$

$$\Rightarrow \| u \|^2 = \int_{-\infty}^{+\infty} x \, dG_{u,u}(x)$$

$$\leq \left(\int_{-\infty}^{+\infty} x \, dG_{T,T}(x) \right) * \left(\int_{-\infty}^{+\infty} x \, dG_{S_\lambda u, S_\lambda u}(x) \right)$$

$$= \| T \|^2 \| S_\lambda u \|^2$$

$$\Rightarrow \| u \| \leq \| T \| \| S_\lambda u \|$$

As $\| T \| \neq 0$, we see that $\| S_\lambda u \| \geq \frac{1}{\|T\|} \| u \|$

$$\Rightarrow \lambda \notin \sigma(S)$$

$\Rightarrow S_\lambda(E)$ is not closed .

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