Endomorphism Ring of GZ-Regular Modules

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Abstract

As a complement to the former work of GZ-regular modules, in this paper along the lines of Z-regular modules due to Zelmanowitz, we improve the study of the endomorphism ring of Z-regular modules to GZ-regular modules. We give a sufficient condition on GZ-regular module M such that $S = End(M)$ is π-regular ring and we prove that R-module M is GZ-regular if and only if $S = End(M)$ is π-regular ring in case that M is a projective finitely power generated R-module. Also we show that for a GZ-regular R-module M, the center of $S = End(M)$, $Cen(S)$, is π-regular ring. Even further if M is a GZ-regular R-module then $R/ann(M)$ is dense in $Cen(S)$.

Keywords: endomorphism ring, Z-regular module, GZ-regular module, π-regular ring, projective module.

حلقة التشاكلات للموديولات المنتظمة من النمط GZ

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الخلاصة

استكمالا للعمل السابق حول الموديولات المنتظمة من النمطGZ، في هذا البحث وعلى غرار الموديولات المنتظمة من النمط Z بحسب زيلمانووتز، نطور دراسة حلقة التشاكلات للموديولات المنتظمة من النمط Z الى الموديولات المنتظمة من النمط GZ. نعطي شرطا كافيا على الموديول M المنتظم من النمط GZ بحيث ان $S = End(M)$ تكون حلقة منتظمة من النمط π ونبرهن ان الموديول M على الحلقة R يكون منتظم من النمط GZ اذا وفقط اذا كانت $S = End(M)$ حلقة منتظمة من النمط π في حالة كون الموديول M هو اسقاطي منتهي قوى التولد على الحلقة R. ايضا نبين ان للموديول المنتظم M من النمط GZ على الحلقة R يكون مركز حلقة التشاكلات $S = End(M)$، $Cen(S)$، هو حلقة منتظمة من النمط π. ابعد من ذلك اذا كان M هو موديول منتظم من النمط GZ على الحلقة R فان $R/ann(M)$ تكون كثيفة في $Cen(S)$.

الكلمات المفتاحية: حلقة تشاكلات، موديول منتظم من النمط Z ، موديول منتظم من النمط GZ ، حلقة منتظمة من النمط π ، موديول اسقاطي.

Introduction

Throughout this paper all rings are commutative with identity and all modules are left unitary, unless otherwise stated. For an R-module M, $S = End(M)$, $T(M)$ and $Cen(S)$ will be denote the endomorphism ring of M, the trace of M and the center of the endomorphism ring $S = End(M)$ respectively. It is well known that a ring R is called regular (in the sense of Von Neumann) if for each $a \in R$, there exists $b \in R$ that $aba = a$ [1]. McCoy in [2] was generalized the concept of regular rings to π-regular rings, a ring R is said to be π-regular if for each $a \in R$ there exist $b \in R$ and a positive integer n such that $a^n b a^n = a^n$. Just like the concept of regular rings was extended to modules in two different ways by Feildhouse[3] and Zelmanowitz[4], the concept of π-regular rings extended to modules in [5] and [6] to two non-equal concepts. Following [7], we denoted Fieldhouse' and Zelmanowitz' regular modules by F-regular and Z-regular modules respectively. Recall that an R-module M is F-regular if each submodule of M is pure [3] and an R-module M is Z-regular if for each $m \in M$ there exists $f \in M^* = Hom\,(M, R)$ such that $f(m)m = m$ [4]. Generalizing these concepts, an R-module M is called GF-regular if for each $x \in M$ and $r \in R$, there exist $t \in R$ and positive integer n such that $r^n t r^n x = r^n x$ [5]. And in a parallel form an R-module M is said to be GZ-regular if for each $x \in M$ and for each $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n f(x) x = r^n x$ for some $f \in M^* = Hom(M, R)$ [6]. According to the latest generalizations, a ring R is π-regular if and only if R is GF-regular R-module if and only if R is GZ-regular R-module [6]. GF-regular modules and GZ-regular modules studied extensively in [5,6,8].Thus, the primary goal of this paper will be to study the endomorphism ring of the GZ-regular module M, $S = End(M)$, and study the relationships between S and a number of concepts like the trace of a module M, $T(M)$, π-regular rings, the center of S, $Cen(S)$ and others. In [4] Zelmanowitz was proved that the endomorphism ring of a Z-regular module need not be Z-regular. Motivated by this fact we investigate whether the endomorphism ring $S = End(M)$ of a GZ-regular R-module M is π-regular or not. Also we show that if M is finitely power generated R-module, then $S = End(M)$ is π-regular ring and we show that if the endomorphism ring of an R-module M, $S = End(M)$, is π-regular then the endomorphism ring of any direct summand of M is π-regular. Moreover for every GZ-regular R-module M the center of the endomorphism ring of M, $Cen(S)$, is π-regular. Furthermore we give examples to explain and support some statements.The rest of the paper is organized as follows. Section 2 is devoted to set up notation and terminology that is not part of the main work. The important of this section is to delineate concise results that are not original but are needed for the paper. Section 3 address the issue of the trace of a GZ-regular R-module M, $T(M)$, and we encountered this issue in connection with the notion of the endomorphism ring of a GZ-regular R-module M, $S = End(M)$, and its center, $Cen(S)$. Finally, in Section 4 we present the main results of our work considering the endomorphism ring of a GZ-regular R-module M, $S = End(M)$, and its relationship with π-regular rings, $T(M)$, $Cen(S)$ and others.

Preliminaries:

In this section we survey some previous not original results of related work which is not part of the technical contribution but is needed in the rest of the paper.

In [6] the author introduced the following definitions.

Definitions 2.1: An R-module M is said to be GZ-regular if for each $m \in M$ and for each $s \in R$ there exist $t \in R$ and a positive integer n such that $s^n t s^n f(m) m = s^n m$ for same $f \in M^* = Hom(M, R)$. Every Z-regular R-module is GZ-regular and a ring R is called GZ-regular if and only if R is GZ-regular as an R-module [6].

Recall that M is GF-regular R-module if for each $m \in M$ and $s \in R$ there exist $t \in R$ and a positive integer n such that $s^n t s^n m = s^n m$ [5].

Every F-regular module is GF-regular and a ring R is GF-regular if R is GF-regular as an R-module [5].

A ring R is π-regular if and only if R is GZ-regular R-module if R is GF-regular R-module [6].

It is well known that any Z-regular module is F-regular, but the converse may not be true in general [7]. Analogously, over any ring a GZ-regular module is GF-regular, but the converse need not be true [6]. The following proposition gives a condition such that the converse true [6].

Proposition 2.2: [6] Let m be an element of GZ-regular R-module M, then for each $s \in R$ there exist $t \in R$ and a positive integer n such that $ann(s^n m) = ann(f(m)s^n t)$ and $f(m)s^n t$ is an idempotent element.

Proposition 2.3: [6] Suppose that M is a projective module over a ring R. M is GZ-regular module if and only if it is GF-regular.

The following concept introduced in [5]:

Definition 2.4: Let P be any submodule of an R-module M. P is said to be G-pure if for each $s \in R$, there exists a positive integer n such that $s^n M \cap P = s^n P$.

Every pure submodule is G-pure [5].

There are many characterizations of GF-regular modules and GZ-regular modules. The following theorem appears in [5].

Theorem 2.5: [5] Let R be any ring. The following statements are equivalent:

(1) M is a GF-regular R-module.

(2) $R/ann(m)$ is a π-regular for each $0 \neq m \in M$.

(3) For each $m \in M$ and $s \in R$ there exist $t \in R$ and a positive integer n such that $s^{n+1}m = s^n m$.

(4) If N is any submodule of M, then N is G-pure.

(5) For each $m \in M$, there exists $u \in R$ and a positive integer n such that $Ru^n m$ is a G-pure submodule.

(6) In case M is finitely generated R-module, then $R/ann(m)$ is a π-regular ring.

The concept of finitely power generated submodule introduced in [6]. Recall that a submodule N of an R-module M is power generated if N is generated by the set $E = \{s_1^{n_1}m_1, s_2^{n_2}m_2, \dots\}$ where $s_i \in R$ and n_i are positive integers. In case E is finite, then N is said to be finitely power generated submodule.

Theorem 2.6: [6] An R-module M is GZ-regular if and only if every power generated submodule of M is projective and direct summed of M.

Corollary 2.7: [6] Every finitely power generated GZ-regular R-module is projective.

Corollary 2.8: [6] Every countable power generated GZ-regular R-module is projective.

The following theorem gives same characterization of GZ-regular modules:

Theorem 2.9:[6] The following conditions are equivalent for any R-module M:

(1) M is GZ-regular module.

(2) For any $m \in M$ and for any $s \in R$ there exists a positive integer n such that the $Rs^n m$ is projective direct summand of M.

(3) For any ${s_i}^{n_i}m_i \in M$ where $m_i \in M$, $s_i \in R$ and n_i are positive integers $i = 1,2,3,\dots.t$ we have that $\sum_{i=1}^{t} R{s_i}^{n_i}m_i$ is projective direct summand of M.

Proposition 2.10: [6] The following conditions are equivalent for any projective R-module M:

(1) M is GZ-regular;

(2) For any $m \in M$ and $s \in R$ there exists a positive integer n such that the $Rs^n m$ is a direct summand of M.

Proposition 2.11: [6] Suppose that M is a projective R-module. M is GZ-regular R-module if and only if every finitely power generated submodule of M is direct summand.

3. The Trace of GZ-regular Modules:

Recall that the trace of an R-module M is :

$$T(M) = \sum_{f \in M^*} f(M)$$

where $M^* = Hom(M, R)$. It's obvious that $T(M)$ is an ideal of R and it is called the trace ideal of M, we denote it by T. It is known that if M is projective module on R, then $T(M) = M$ and T is pure ideal [6], which implies that T is G-pure ideal, also $ann(T) = ann(M)T$ [8].

The following proposition shows that the trace of GZ-regular modules satisfies the same properties without assuming the module to be projective.

Proposition 3.1: Let M be a GZ-regular R-module then:

(1) $T(M) = M$.

(2) T is G-pure ideal in R can be generated by idempotent elements.

(3) $ann(T) = ann(M)$.

Proof:

(1) It is clear that $MT \subseteq M$ and M is GZ-regular module, then for each $m \in M$ and for each $r \in R$, there exist $t \in R$ and a positive integer n such that $r^n t r^n f(m) m = r^n m$ for some $f \in M^* = Hom(M, R)$. But $f(m) \in T$, therefore $r^n m \in MT$. Take r=1 we get that $m \in MT$ and hence $M \subseteq MT$ which implies that $MT = M$.

(2) It is clear that $r^m T \subseteq T \cap r^m R$ for each $r \in R$ and for each positive integer m. Now let $t \in T \cap r^m R$, then $t \in T$ and $t = r^m y$ for some $y \in R$. We have to prove that $t \in r^m T$. Since $t \in T$, then $t = \sum_{i=1}^{n} h_i(x_i)$ where $h_i \in M^*$ and $x_i \in M$. Because M is GZ-regular R-module, so for each $x_i \in M$ and for each $r \in R$, there exist s $\in R$ and a positive integer k such that $r^k s r^k f(x_i) x_i = r^k x_i$ for some $f_i \in M^* = Hom(M, R)$. Therefore $r^k t = \sum_{i=1}^{n} h(r^k x_i) = \sum_{i=1}^{n} h_i(r^k s r^k f_i(x_i) x_i)$.

Put $w_i = f_i(x_i) r^k s$ givs $w_i = w_i^2$ and $r^k w_i = w_i r^k x_i$[6, Proposition 3.2.3]. Since this is true for each $r \in R$, then $W = 1 - \prod_{i=1}^{n} 1 - w_i \in T$.

Thus $Wx_i = Rx_i$, so $t = r^k t$ and hence $t \in r^k T$ which means that $T \cap r^k T \subseteq r^k T$. Consequently, $r^k T = T \cap r^k T$ and this means that T is a G-pure ideal.

Now to prove the other part, let $t \in T$, then $t = \sum_{i=1}^{n} h_i(x_i)$ where $h_i \in M^*$ and $x_i \in M$. Since M is GZ-regular R-module then there exist $f_i \in M^*$ such that $r^k s r^k f(x_i) x_i = r^k x_i$, i=1,2,3....n. As $w_i = w_i^2$ for each $r \in R$, then $t = \sum_{i=1}^{n} h_i(x_i)$ and T can be generated by an idempotent element.

(3) By (1) $M = MT$, then $ann_R(T) \subseteq ann_R(M)$. On the other hand let $t \in T$, hence $t = \sum_{i=1}^{n} h_i(x_i)$. Suppose that $r \in ann_R(M)$, so $rt = \sum_{i=1}^{n} h_i(rx_i) = 0$. Therefore $r \in ann(T)$ and $ann(T) = ann(M)$. ■

4- The Endomorphism Ring of GZ-Regular Modules:

In this section we investigate the relationship between a GZ-regular R-module M and its endomorphism ring $S = End(M)$ and we seek answers to the questions:

(1) If M is GZ-regular module, is $S = End(M)$ a π- regular ring? And

(2) When does the converse of (1) become true?

Also we investigate the answers for the questions with respect to the center of the endomorphism ring of M which we denoted by $Cen(S)$.

Lemma 4.1: Let M be an R-module. The endomorphism ring S is a π-regular if and only if for each $f \in S$ there exists a positive integer n such that $Ker(f^n)$ and $Im(f^n)$ are direct summands of M.

Proof: Let S be a π-regular endomorphism ring, then for each $f \in S$ there exist $g \in S$ and a positive integer n such that $f^n g\, f^n = f^n$. Therefore by [10, Theorem 8, p: 23] there exist $g|Im(f^n)$ splits the exact sequence.

$$0 \to Ker(f^n) \to Ker(f^n) \oplus Im(f^n) \to Im(f^n) \to 0$$

and gf^n splits the exact sequence

$$0 \to Im(f^n) \to Ker(f^n) \oplus Im(f^n).$$

This means that the following diagram is commutative [11].

$$\begin{array}{ccccccccc} 0 & \to & Ker(f^n) & \longrightarrow & M & \longrightarrow & Im(f^n) & \to & 0 \\ & & \simeq\updownarrow id & & \simeq\downarrow & & \simeq\updownarrow id & & \\ 0 & \to & Ker(f^n) & \underset{i_{ker(f^n)}}{\longrightarrow} & Ker(f^n) \oplus Im(f^n) & \underset{\pi_{Im(f^n)}}{\longrightarrow} & Im(f^n) & \to & 0 \end{array}$$

which implies that $Ker(f^n)$ and $Im(f^n)$ are direct summand of M.

Conversely, suppose that for each $f \in S$, there exists a positive integer n such that $Ker(f^n)$ and $Im(f^n)$ is direct summand of M. Then there exists $\acute{g}, Im(f^n) \to M$ such that $y\acute{g}f^n = y$ for each $y \in Im(f^n)$, that is $xf^n\acute{g}f^n = xf^n$ for each $x \in M$. But $Im(f^n)$ is a direct summand of M, so we can extend $\acute{g}$ to g on M by taking $g = 0$, on the other supplement of the summand. Therefore, for any $x \in M$, $xf^ngf^n = xf^n$ which implies that $f^ngf^n = f^n$. Thus S is π-regular ring. ■

The following example shows that there is a GZ-regular module $M = P \oplus N$ such that the endomorphism ring of P and the endomorphism ring of N are π-regular rings, but $M^* = End(M)$ is not π-regular.

<u>Example 4.2:</u> Let F be field. Put $F_i = F$, $i \in I$. Let $R = \prod_{i=1}^{\infty} F_i$. Let $P = R$ and $N = \bigoplus_{i \in I} F_i \subset \prod_{i=1}^{\infty} F_i = R$. Let $M = P \oplus N$, then P, N and M are Z-regular module [11] and hence, P, N and M are GZ-regular. Again by [11] $End(P) \simeq R \simeq End(N)$ are GZ-regular rings, so π-regular rings. Now we claim that $End(M)$ is not π-regular ring. For, let f be the endomorphism of M defined such that $f^n: (x_1, x_2) \to (x_2, 0)$ for same positive integer n, therefore $Im(f^n) = \bigoplus_{i \in I} F_i \subset R = P$. Since $Im(f^n) = \oplus_{i \in I} F_i$ is not a direct summand of R, then $Im(f^n) = \bigoplus_{i \in I} F_i$ is not a direct summand of $M = P \oplus N$ [11]. Consequently, by Lemma 4.1 $End(M)$ is not π-regular ring. ■

Concerning question (1) we do have the following:

Proposition 4.3: If M is a GZ-regular finitely power generated R-module, then $End(M)$ is π-regular ring.

Proof: Let f be an endomorphism of M of such that there exists a positive integer n with $f^n \in End(M)$. Since M is finitely power generated, then $Im(f^n)$ is finitely power generated submodule of M. Because M is GZ-regular, so by [6, Theorem 3.2] $Im(f^n)$ is projective and direct summand of M. Now we have the exact sequence

$$0 \to Ker(f^n) \to M \to Im(f^n) \to 0$$

Since $Im(f^n)$ is projective, therefore by [10, Theorem 1.2.8] the above sequence is split and this means that $M = Im(f^n) \oplus Ker(f^n)$. Thus $End(M)$ is π-regular ring by Lemma 4.1.■

It is well know that if M is an R-module, N is a direct summand of M and π is the projection of M onto N, then π is an idempotent of $S = End(M)$ and $End(N) = \pi S \pi$. Also it is known that if R is π-regular ring and $x \in R$, then there exist $y \in R$ and a positive integer n such that $x^n = x^n y x^n$, by taking $e = x^n y$ we get that e is an idempotent element satisfies $Re = Rx^n$.

Lemma 4.4: Let M be an R-module and N be any direct summand of M. If $End(M)$ is π-regular ring, then $End(N)$ is π-regular ring.

Proof: Let N be any direct summand M and π be the projection of M onto N. We that that $End(N) = \pi S \pi$, but $\pi S \pi$ is π-regular ring for any π-regular ring S and any idempotent $\pi \in S$, therefore $End(N)$ is π-regular ring.■

To answer question 2 we give conditions such that the converse of Proposition 4.3 is true.

Theorem 4.5: Let M be a projective finitely power generated R-module. M is GZ-regular module if and only if $End(M)$ is π-regular ring.

Proof: If M is a GZ-regular module, then $End(M)$ is π-regular ring by Proposition 3.4. Conversely, suppose that $End(M)$ is a π-regular module. If M is cyclic module and since M is projective finitely power generated, then for each $r \in R$ there exists $x \in R$ such that $M \simeq$

$Rr^n x$ for some positive integer n such that $r^n x = e = e^2 \in R$, hence, it is clear that $End(M) \simeq Re$. Since $End(M)$ is π-regular ring, therefore $End(M)$ is GZ-regular R-module [5]. Thus Re is GZ-regular module which implies that M is a GZ-regular module. Now for any module M, by the dual basis Lemma, let $\{r_i^{n_i} x_i\}_{i\in I}$ be a generating set of M and $\{f_i\}_{i\in I} \subset End(M)$ such that for each $x \in M$ we have that $f_i(x) = 0$ for all but finite number of i and $x = \sum_{i\in I} f_i(x)\, r_i^{n_i} x_i$. Define the map $P_i: M \to Rr_i^{n_i} x_i$ by $P_i(x) = f_i(x) r_i^{n_i} x_i$ for each $x \in M$. Therefore P_i is an endomorphism of M. Since $End(M)$ is π-regular ring, then by Lemma 4.1 we get that $P_i(M)$ is a direct summand of M and hence, a direct summand of $Rr_i^{n_i} x_i$, so $P_i(M)$ is cyclic and by Lemma 4.4 the endomorphism of $P_i(M)$ is π-regular. Hence, by the first part of this theorem we conclude that $P_i(M)$ is a GZ-regular module. Now by [6, Proposition 3.4.2] and [5, Corollary 24] the module $\bigoplus_{i \in I} P_i(M)$ is GZ-regular. However, $M = \sum_{i\in I} P_i(M)$, so it is clear that there is a natural epimorphism $\bigoplus_{i \in I} P_i(M) \to \sum_{i\in I} P_i(M) \to 0$. Since $M = \sum_{i\in I} P_i(M)$ is projective, then M is direct summand of $\bigoplus_{i \in I} P_i(M)$ [10, Theorem 1.28, P:23], but $\bigoplus_{i \in I} P_i(M)$ is GZ-regular module wherefore M is GZ-regular module again by [6, Proposition 3.4.2] and [5, Corollary 24]. ■

In the rest of this section we study the center of the endomorphism ring $S = End(M)$, which we denoted by $Cen(S)$. We showed in example 4.2 that the endomorphism ring of a GZ-regular module need not be π-regular ring. However, in the following proposition we prove that $Cen(S)$ is π-regular ring.

Proposition 4.6: Let M be an R-module. If M is GZ-regular, then $Cen(S)$ is π-regular ring.

Proof: Let $f \in Cen(S)$, since M is GZ-regular module then for each $x \in M$, there exist $g \in M^* = Hom(M, R)$ and a positive integer n such that

$$w^n(x) v(x) w^n(x) g\big(f(x)\big) f(x) = w^n(x) f(x)$$

where $w, v \in M^* = Hom(M, R)$. But since $f \in Cen(S)$, therefore

$$w^n(x)f(x) = w^n(x)v(x)w^n(x)g\big(f(x)\big)f(x) = w^n(x)v(x)w^n(x)f\big(g(x)\big)f(x) = w^n(x)v(x)w^n(x)f\big(f(x)\big)g(x) = w^n(x)v(x)w^n(x)f^2(x)g(x).$$

Now we can write $w^n(x)x$ as the following:

$$w^n(x)x = w^n(x)v(x)w^n(x)f(x)g(x) + [w^n(x)x - w^n(x)v(x)w^n(x)f(x)g(x)].$$

It is clear that

$$w^n(x)v(x)w^n(x)f(x)g(x) \in Im(f).$$

Now

$$f(w^n(x)x - w^n(x)v(x)w^n(x)f(x)g(x)) = f(w^n(x)x) - w^n(x)v(x)w^n(x)f^2(x)g(x) = f(w^n(x)x - w^n(x)f(x)) = w^n(x)f(x) - w^n(x)f(x) = 0.$$

Thus

$$M = Im(f) + Ker(f),$$

but if $f(x) \in Im(f) \cap Ker(f)$, then $f(x) \in Ker(f)$, which implies that $f\big(f(x)\big) = f^2(x) = 0$, and hence $f(x) = 0$. Thus $M = Im(f) \oplus Ker(f)$, so $Cent(S)$ is π-regular ring by Lemma 4.1. ■

Remark 4.7: Let M be an R-module, define a map $f: R \to End(M)$ by $f(r) = f_r$ where $f_r(x) = rx$ for all $x \in M$. It is clear that f is a ring homomorphism and that $Ker(f) = ann(M)$. Hence by f we can consetruct a ring homomorphism g: $R/ann(M) \to End(M)$ such that $g\big(r + ann(M)\big) = f(r)$ and g is a monomorphism, consequently it can be consider $R/ann(M)$ as a subring of $End(M)$ and moreover $\bar{R} = R/ann(M) \subseteq Cen(s)$.

We mention that if $f: M \to R$ is homomorphism, then for each $y \in M$ define the homomorphism, $f_y: M \to M$ by $f_y(x) = y\,f(x)$ for all $x \in M$ [13].

The following proposition and its corollaries appears in [12]

Proposition 4.8: Let M be an R-module. If $T(M)$ is generated by an idempotent and $ann(M) = ann(T)$, then $Cen(S) = R/ann(M)$.

Proof: Let $g \in Cen(S)$, then $g \circ f_y = f_y \circ g$ for each $y \in M$, where $f_y(x) = yf(x)$ for each $x \in M$ as defined above. Thus $g\ o\ f_y(x) = f_y\ o\ g(x)$, that is $g\left(f_y(x)\right) = f_y(g(x))$, which means that $g\left(y\left(f(x)\right)\right) = yf(g(x))$. Hence $f(x)g(y) = y.f\left(g(x)\right) \ldots$ (*). Since $T(M)$ is generated by an idempotent $e = e^2 \in T(M)$, then $e = \sum_{i\in I} f_i(x_i)$, where $x_i \in M$, and $f_i \in M^* = Hom(M, R)$. Therefore $e.g(y) = \sum_{i\in I} f_i(x_i)\, g(y)$ for each $y \in M$, so that by (*) $g(e.y) = \sum_{i\in I} f_i(x_i)\, y$. Because $ann(M) = ann(T(M))$, thus $e.g(y) = g(y)$, which implies that $g(y) = ry = \varphi_r y$ where $r = \sum_{i\in I} f_i(g(x_i)) \in R$. Consequently $g \in R/ann(M)$ and hence $Cen(S) \subseteq R/ann(M)$. Therefore $Cen(S) = R/ann(M)$. ■

Corollary 4.9: Let M be an R-module such that $T(M) = R$, then $Cen(S) = R$.

Proof: Since $ann(M) \subseteq ann(T(M))$ for any R-module M and since $ann\left(T(M)\right) = 0$, then $ann(M) = ann\left(T(M)\right) = 0$. Hence, by Proposition 4.8 we have that $Cen(S) = R$. ■

Corollary 4.10: Let M be an R-moudle. If M is finitely generated projective module, then $Cen(S) = R/ann(M)$.

Proof: Since M is finitely generated projective module, then $T(M)$ is finitely generated [9]. Also since M is projective, then $T(M)$ is pure and $ann\left(T(M)\right) = ann(M)$ [14]. Now we have that $T(M)$ is pure and finitely generated, so $T(M)$ is generated by an idempotent [9]. Therefore by Proposition 4.8 we have that $Cen(S) = R/ann(M)$. ■

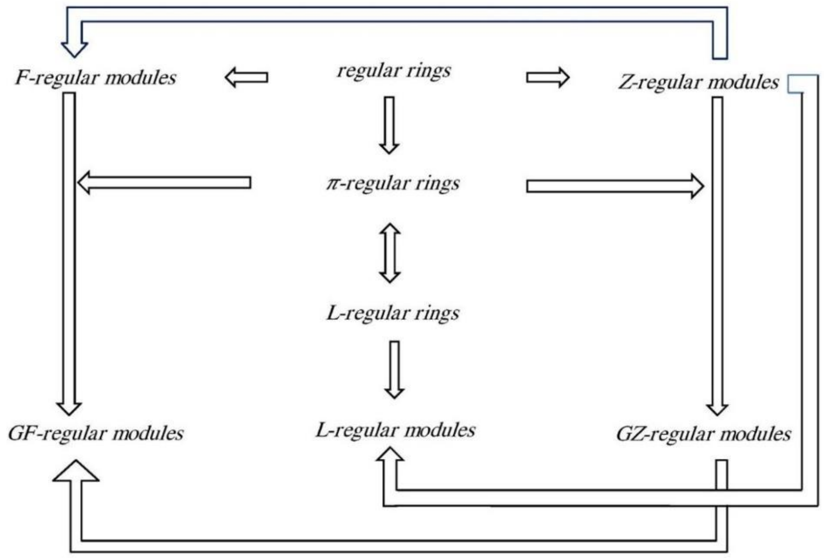
With the following definition which appeared in [15] we can use GZ-regular modules to study the behavior of $R/ann(M)$ in $Cen(S)$.

Definition 4.11: A subset D of $S = End(M)$ is said to be dense in S if for every finite set $\{x_1, \ldots \ldots x_n\}$ of elements of M and any $\alpha \in S$, there exists $\delta \in D$ such that $\alpha(x_i) = \delta(x_i)$ for $1 \le i \le n$.

Theorem 4.12: Let M be an R-module. If M is a GZ-regular module, then $R/ann(M)$ is dense in $Cen(S)$.

Proof: Let N be a submodule of M generated by the set $\{x_1, \dots \dots x_n\}$ of M. Since M is a GZ-regular, then by [6, Theorem 3.3.2] there exists a projective submodule K of M such that $M = N \oplus K$. Let $\beta: M \to M$ be a homomorphism defined by $\beta(n,k) = (n,0)$ for each $n \in N, k \in K$ and let $f \in Cen(S)$, accordingly for $\beta = \beta\ o\ f$. Put $f_N = f|N$ then $f_N \in End(N) = \acute{S}$ (for if $f_N: N \to M$ but $f\ o\ \beta(M) = \beta\ o\ f(M) = \beta(f(M)) \subseteq N$, thus $f_N: N \to N$). We have to prove that $f_N \in Cen(\acute{S})$, let $g \in \acute{S}$, then g can be extended to an endomorphism of M, defined by $P(n,k) = (g(n),k)$ for each $n \in N, k \in K$. Therefore $f \circ \rho = \rho \circ f$ which implies that $g \circ f_N = f_N \circ g$. Since N is projective by [6, Theorem 3.3.2] and finitely generated, then by corollary 4.10 there exists $r \in R$, such that $f(x_i) = f_N(x_i) = rx_i$ where $1 \leq i \leq n$. There fore $R/ann(M)$ is dense in $Cen(S)$. ■

Recall that an element m in an R-module M is L-regular if there exists $\alpha \in M^* = Hom_R(M,R)$ such that $(\alpha(m))^2 = \alpha(m)$ and $m - \alpha(m)m \in L(M)$. An R-module M is L-regular if each element of M is L-regular. A ring R is L-regular if R is L-regular as an R-module [16]. The following diagram shows all implications among these properties (with no other implications holding, except by transitivity):



Conclusion

In this work we study and investigate the relationship between GZ-regular R-module M and their endomorphism rings $S = End(M)$. In particular we describe GZ-regular modules whose endomorphism rings are π- regular. Moreover if M be a projective finitely power generated R-module, then M is GZ-regular module if and only if $End(M)$ is π-regular ring. On the other hand we proved that if M is GZ-regular module, then $Cen(S)$ is π-regular ring. Finely we exploited GZ-regular modules to study the behavior of $R/ann(M)$ in $Cen(S)$ such that if M is a GZ-regular module, then $R/ann(M)$ is dense in $Cen(S)$.

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